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MATHEMATICS

Approximate Method of Calculating Characteristic Elements and Values

by

T. LEŻAŃSKI

Presented by S. MAZUR on September 3, 1956

This note presents a new method of approximate calculation of elements and characteristic values of a bounded operation, symmetrical in the Hilbert space.

The methods hitherto used for this purpose are based on the proposition that $\inf(Ax,x)$ (or $\sup(Ax,x)$) under the condition $\|x\|^2=1$ is the least (or, respectively, the greatest) characteristic value of operation A, and the element realising the above extremum is the characteristic element corresponding to the characteristic value in question.

If the least characteristic value λ_1 is known, then in order to calculate the next (assuming that in the interval in question operation A has a point spectrum) we must calculate $\inf(Ax,x)$ under the condition that $\|x\|^2=1$ and $(x,h)=0$ ($h \in G_{\lambda_1}$), where G_{λ_1} is the characteristic space for the characteristic value λ_1. Even if the characteristic spaces are one-dimensional, the calculation of λ_2 is charged with the error of calculation of the characteristic element x_1 for the characteristic value λ_1; in passing to the further values and characteristic elements these errors are summed. The situation becomes still more complicated if the spaces $G_{\lambda_1}, G_{\lambda_2}, \ldots$ have many or infinitely many dimensions. These difficulties are avoided in the method given below, which reduces the approximate calculation of the characteristic value λ_Δ, contained in the given interval Δ, and of the orthogonal projection of a given element z_0 upon G_Δ, the characteristic space corresponding to the characteristic value λ_Δ, to the calculation of the conditional minimum of a certain quadratic functional.

Notations, definitions and assumptions

H — the complex Hilbert space;

A — a bounded linear symmetrical operation from H to H;

$\alpha_1 < \alpha < \beta < \beta_1$ — real numbers;

$\widetilde{H}$ — the Hilbert space of "abstract functions" $\{x_\lambda\}$, $x_\lambda \in H$;

$\alpha \leqslant \lambda \leqslant \beta$, with the scalar product $[\{x_\lambda\}, \{y_\lambda\}] \overset{\mathrm{df}}{=} \int\limits_\alpha^\beta (x_\lambda, y_\lambda)\, d\lambda$;

$\widetilde{M}$ — the linear set of functions $\{x_\lambda\}$ in $\widetilde{H}$, such that the integral $\int\limits_\alpha^\beta x_\lambda . d_\lambda$ exists and $\in H$;

λ_Δ — the unique characteristic value of operation A in the interval $\langle \alpha, \beta \rangle$;

G_Δ — the characteristic space of operation A for the characteristic value λ_Δ;

$z_{(\Delta)}$ — the orthogonal projection of element z on G_Δ.

It is assumed that the spectrum of operation A in the interval $\langle \alpha, \beta \rangle$ is a point spectrum and that in $\langle \alpha_1, \alpha \rangle$, $\langle \beta, \beta_1 \rangle$ operation A has no characteristic values.

Results

We pose ourselves the following problem:

1) Approximate calculation of the characteristic value λ_Δ.

2) Approximate calculation for a given element $z_0 \in H$, of the orthogonal projection $z_{0_{(\Delta)}}$ of z_0 on G_Δ.

Let

$$\vartheta = \max\left[1, \left|\frac{\alpha-\beta}{\alpha_1-\alpha}\right|, \left|\frac{\alpha-\beta}{\beta_1-\beta}\right|\right], \bar{\delta} = \min\left[|\alpha_1-\alpha_1|, |\beta_1-\beta|\right];$$

then we obtain three inequalities:

$$\text{(I)} \qquad |\lambda-\lambda_\Delta|\,\|x\| \leqslant \vartheta \|Ax-\lambda x\| \qquad (\alpha \leqslant \lambda \leqslant \beta);$$

$$\text{(II)} \qquad \bar{\delta}\|x-x_{(\Delta)}\| \leqslant \|Ax-\lambda_\Delta x\|;$$

$$\text{(III)} \qquad \big|\|Ax-\lambda x\| - \|Ax-\lambda_\Delta x\|\big| \leqslant |\lambda-\lambda_\Delta| \cdot \|x\| \qquad (\alpha \leqslant \lambda \leqslant \beta).$$

LEMMA 1.

$$\int\limits_\alpha^\beta \|x_\lambda - x_{\lambda_{(\Delta)}}\|^2 d\lambda \leqslant \left(\frac{1+\vartheta}{\bar{\delta}}\right)^2 \Phi(\{x_\lambda\}) \quad \textit{where} \quad \Phi(\{x_\lambda\}) = \int\limits_\alpha^\beta \|Ax_\lambda - \lambda x_\lambda\|^2 d\lambda.$$

LEMMA 2. *For a given fixed function $\{x_\lambda\} \in \widetilde{M}$, we have the inequality*

$$\left\|\int\limits_\alpha^\beta x_\lambda d_\lambda\right\|^2 \leqslant (\beta-\alpha)\int\limits_\alpha^\beta \|x_\lambda\|^2 d\lambda.$$

Let $\{\bar{x}_\lambda\} \in \widetilde{M}$ be an arbitrarily fixed function.

Let $\bar{x} = \int\limits_\alpha^\beta \bar{x}_\lambda d_\lambda$, $\bar{\lambda} = (A\bar{x}, \bar{x}) \cdot \|\bar{x}\|^{-2}$. We have, of course, $\bar{x}_{(\Delta)} = \int\limits_\alpha^\beta \bar{x}_{\lambda_{(\Delta)}} d_\lambda$ since the operation of projecting is linear and bounded.

Lemmas 1 and 2 imply

LEMMA 3. $\|\bar{x}-\bar{x}_{(\Delta)}\|^2 \leqslant (\beta-\alpha)\left(\dfrac{1+\vartheta}{\bar{\delta}}\right)^2 \Phi(\{x_\lambda\})$.

LEMMA 4. $\|A\bar{x}-\bar{\lambda}x\|^2 < (\beta-\alpha)(1+\vartheta)^2\Phi(\{x_\lambda\})$.

With the previous definitions of element $\bar{x}$ and number $\bar{\lambda}$, let $\varrho=(\beta-\alpha)^{1/2}(1+\vartheta)\big(\Phi(\{\bar{x}_\lambda\})\big)^{1/2}$. We have for $\{\bar{x}_\lambda\}$ such that $\|\bar{x}\|\delta-\varrho>0$.

THEOREM 1. $|\bar{\lambda}-\lambda_\Delta| \leqslant \dfrac{\varrho\cdot\bar{\delta}}{\|\bar{x}\|\cdot\bar{\delta}-\varrho}$.

For a given number $\varepsilon>0$ we shall define a set Z_ε in the space H as the set of elements of the form $z=\int\limits_\alpha^\beta x_\lambda d_\lambda$, where $\{x_\lambda\}\in\widetilde{M}$ and $\Phi(\{x_\lambda\})<\varepsilon$. We have $\{x_\lambda\}\in\widetilde{M}$ and $\Phi(\{x_\lambda\})<\varepsilon$.

LEMMA 5. *If* $z\in\bar{Z}_\varepsilon$, $h\in G_{(\Delta)}$, *then* $z+h\in\bar{Z}_\varepsilon$.

Lemma 5 implies

THEOREM 2. *If*

$$\Big\|z_0-\int\limits_\alpha^\beta \bar{x}_\lambda d_\lambda\Big\|^2-\inf\Big\{\Big\|z_0-\int\limits_\alpha^\beta y_\lambda d_\lambda\Big\|^2 : \Phi(\{y_\lambda\})<\varepsilon_1\Big\}\leqslant\varepsilon_2,$$

then

$$\|\bar{x}-z_{0_{(\Delta)}}\| \leqslant \varepsilon_1^{1/2}\cdot(\beta-\alpha)^{1/2}\cdot\frac{1+\vartheta}{\bar{\delta}}+2\varepsilon_2.$$

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MATHEMATICS

Approximate Solution of a Linear Equation

by

T. LEŻAŃSKI

Presented by S. MAZUR on September 3, 1956

This paper contains a new method of the approximate solution of the linear equation $Ax=\varphi_0$, where the solution x and the right-hand side φ_0 belong to the Hilbert spaces H and H_1 respectively, and A is a linear bounded operation from H to H_1.

The essentials of this method are t-real "abstract functions" $x(t)$ with values in H, concerning which it is proved that $\|Ax(t)-\varphi_0\|_1\to 0$, $(t\to\infty)$. It is also proved that if the equation $Ax=\varphi_0$ has a unique solution x^*, then $x(t)\to x^*(t\to\infty)$. A similar method was published earlier by L. Kantorowicz [1], in which he did not use the curve $x(t)$ but the sequence of elements $x_1, x_2, \ldots$, and obtained a similar estimation of the expression $\|Ax_n-\varphi_0\|_1$ with respect to convergence. In the general case, however, i. e., where the only condition is the boundedness of operation A, he did not prove the convergence of the sequence x_n to the solution x^*.

Notations, definitions and assumptions

H, H_1 — real Hilbert spaces whose elements will be denoted by $x, y, \ldots$ and φ, ψ respectively, scalar products by (x,y) and $(\varphi,\psi)_1$ and norms by $\|x\|$ and $\|\varphi\|_1$.

A — a linear bounded operation from H to H_1.

φ_0 — a fixed element of space H_1.

We assume that the equation

$$Ax=\varphi_0$$

has exactly one solution $x^*\in H$.

$\bar{A}$ — an operation conjugate to A. $\bar{A}$ is a linear bounded operation from H_1 to H. Let $z_0=\bar{A}\varphi_0$; we have $z_0\in H$.

$B = \bar{A}A$. B is a linear symmetrical operation from H to H. For a linear bounded U-operation from H to H let

$$e^{U} = I + \frac{1}{1!}U + \frac{1}{2!}U^2 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}U^n,$$

and then for a given $x_0 \in H$

$$x(t) = e^{-tB}(x_0) + \int_0^t e^{-sB}(z_0)\,ds. \tag{F}$$

Results

THEOREM 1. $\|Ax(t)-\varphi_0\|_1^2 \leqslant (2te)^{-1}\cdot\|x^*-x_0\|^2$.

The above theorem makes it (theoretically) possible to approximate the solution x^* by the element x in the sense that the number $\|Ax-\varphi_0\|_1 = \|A(x-x^*)\|_1$ may be as small as we like. We also have

THEOREM 2. *Function $x(t)$ defined by formula* (F) *has the property*

$$\|x(t)-x^*\| \to 0 \; (t\to\infty),$$

but without additional assumptions we have no estimation of $\|x(t)-x^*\|$.

THEOREM 3. *If a number $m>0$ satisfies $(x,x) \leqslant m(Bx,x)$, then*

$$\|x(t)-x\|^2 \leqslant e^{-2tm}\|x^*-x_0\|^2.$$

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MATHEMATICS

Approximate Calculation of the Minimum of a Convex Functional

by
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Presented by S. MAZUR on September 3, 1956

The object of this note is to give a new method of the approximate calculation of the minimum of a convex functional defined in the Hilbert space.

This method is a generalisation of the method used in the preceding paper *Approximate Solution of a Linear Equation*, where we calculated in an approximate manner the minimum of the quadratic convex functional $\|Ax-\varphi_0\|_1^2$ by means of "abstract functions" $x(t)$ satisfying the differential equation $x'(t)=-\bar{A}(Ax-\varphi_0)$. It is now easy to verify that the expression $2\bar{A}(Ax-\varphi_0)$ is a gradient of the functional $\|Ax-\varphi_0\|_1^2$ the minimum of which we wish to calculate in the paper mentioned above.

This remark brings us in the general case of a convex functional $\Phi(x)$ to the consideration of functions $x(t)$ satisfying the differential equation $x'(t)=\text{gradient } \Phi(x(t))$; we find that for such functions $x(t)$ and $t\to\infty$ we have $\Phi(x(t))\to\min\Phi(x)\,(x\in H)$. We do not know, however, whether $x(t)$ has a limit for $t\to\infty$.

Notations, definitions and assumptions

H — the real Hilbert space the elements of which will be denoted by $x,y,\ldots,z$ with the scalar product (x,y) and the norm $\|x\|=(x,x)$.

$\Phi(x)$ — a functional defined on H, real, continuous and bounded from below. It is assumed that

1) the functional $\Phi(x)$ is convex, i. e., satisfies the inequality

$$\Phi(\alpha x+\beta y)\leqslant\alpha\Phi(x)+\beta\Phi(y)\qquad\begin{pmatrix}\alpha\geqslant 0,\ \beta\geqslant 0\\ \alpha+\beta=1\end{pmatrix},$$

2) at each point $x \in H$ except, possibly, the points at which the functional $\Phi(x)$ has its infimum, this functional has a gradient which will be denoted by grad $\Phi(x)$, i. e., an element satisfying the relation

$$\Phi(x+h)-\Phi(x)=(\operatorname{grad}\Phi(x), h)+R(x,h),$$

where

$$R(x,h)=O(\|h\|).$$

It is assumed, moreover, that for each point x_0, except those points at which $\Phi(x)$ has its infimum, there exists an "abstract function" $x(t)$, $x(t) \in H$, t being real, which satisfies the differential equation

$$\text{(G)} \qquad x'(t)=-\operatorname{grad}\Phi\big(x(t)\big)$$

and the initial condition $x(0)=x_0$.

Results

LEMMA 1. *If function $y(t)$ has a continuous derivative $y'(t)$, then*

$$\frac{d}{dt}\Phi\big(y(t)\big)=\big(\operatorname{grad}\Phi(y(t)), y'(t)\big).$$

COROLLARY. If function $x(t)$ satisfies (G), then the real function $\Phi\big(x(t)\big)$ is non-increasing, since $\frac{d}{dt}\Phi\big(x(t)\big)=-\|\operatorname{grad}\Phi\big(x(t)\big)\|^2$.

LEMMA 2. $\Phi(x)+\big(\operatorname{grad}\Phi(x), y-x\big)\leqslant\Phi(y) \quad (x,y \in H)$.

LEMMA 3. *If $\Phi(x_2)\leqslant\Phi(x_1)$, then* $(\operatorname{grad}\Phi(x_1), x_1-x_2)\geqslant 0$.

LEMMA 4. *If function $x(t)$ satisfies* (G) *and if for $\bar{x} \in H$ we have the inequality $\Phi(\bar{x})\leqslant\Phi\big(x(t)\big)$ $(t_1\leqslant t\leqslant t_2)$, then function $\|x(t)-\bar{x}\|$ is non-increasing in the interval $\langle t_1, t_2\rangle$.*

DEFINITION 1. *For $x \in H$ let $\delta(x)=$ the infimum of numbers δ satisfying the condition* $\inf\{\Phi(y) : \|x-y\|\leqslant\delta\}=\inf\{\Phi(y) : y \in H\}$.

We shall assume that $\delta(x)<+\infty$ for each $x \in H$.

LEMMA 5. *If function $x(t)$ satisfies* (G), *then $t_2>t_1$ implies $\delta\big(x(t_2)\big)\leqslant\delta\big(x(t_1)\big)$.*

LEMMA 6. *If $\delta(x_0)>0$, then* $\|\operatorname{grad}\Phi(x_0)\|\geqslant\{\Phi(x_0)-\inf\Phi(x)\}\dfrac{1}{\delta(x_0)}$.

THEOREM 1. *If function $x(t)$ satisfies* (G) *and if for t_0 we have $\delta\big(x(t_0)\big)>0$, then*

$$\Phi\big(x(t)\big)-\inf\Phi(x)\leqslant\left\{\frac{t-t_0}{\delta\big(x(t_0)\big)}-\frac{1}{\Phi\big(x(t_0)\big)-\inf\Phi(x)}\right\}^{-1}.$$

Theorem 1 makes it possible (theoretically) to find the element x realising the minimum of functional $\Phi(x)$ in an approximate manner in the sense that the number $|\Phi(x) - \min\Phi|$ may be as small as we like; it is not known whether $x(t)$ has a limit.

We have, however,

THEOREM 2. *If function $x(t)$ satisfies the equation*

$$x'(t) = -\operatorname{grad}\Phi(x(t)),$$

and if the functional $\delta(x)$ satisfies the inequality

$$\delta(x) \leqslant f(\Phi(x) - \min\Phi),$$

where the non-negative continuous function $f(t)$ is such that $\lim\limits_{\varepsilon\to+0}\int\limits_{\varepsilon}^{a} sf(s)ds < +\infty$, *then there exists an $x^* \in H$ such that* $\|x(t) - x^*\| \leqslant \int\limits_{0}^{\Psi(t)} s^{-1}f(s)\,ds$, *where* $\Psi(t) \overset{\mathrm{df}}{=} (x(t)) - \min\Phi$.

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MATHEMATICS

Bases in Certain Spaces of Continuous Functions

by

C. BESSAGA

Presented by S. MAZUR on September 26, 1956

Let the symbol $C(Q/H)$ denote the Banach space of all real functions $x(t)$ defined and continuous on a compact metric space Q and equal to zero on a closed subset $H \subset Q$ with the norm

$$\|x\| = \sup_{t \in Q} |x(t)|.$$

(If H is the empty set Λ, then instead of writing $C(Q/\Lambda)$ we shall briefly write $C(Q)$).

In 1955 F. S. Vakher published a paper [3] in which she proved the following theorem:

THEOREM W. *In the space $C(Q)$ there exists a basis in the sense of Schauder* *).

The result presented in this paper was obtained at the same time **). The following theorem holds:

THEOREM 1. *If D is a Hilbert cube, i. e., if*

$$D = \underset{t=(\tau_n)}{\mathcal{E}} \left[0 \leqslant \tau_n \leqslant \frac{1}{n} \quad (n=1,2,\ldots) \right], \tag{1}$$

or if D is an n-dimensional cube $(n=1,2,\ldots)$, then a basis with the norm 1 exists in the space $C(D/H)$.

The norm of the basis (e_n) of the Banach space is the least positive number K for which the condition

$$\|t_1 e_1 + t_2 e_2 + \ldots + t_p e_p\| \leqslant K \|t_1 e_1 + t_2 e_2 + \ldots + t_q e_q\| \tag{2}$$

is satisfied for arbitrary real values $t_1, t_2, \ldots, t_q$ and for arbitrary natural $p < q$.

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*) For the definition and fundamental properties of the Schauder basis see [1].

**) This result was presented at a meeting of the Polish Mathematical Society in December 1954. The present publication has been delayed for reasons beyond the author's control.

(x) *The sequence* (e_n) *of elements of a Banach space whose linear combinations constitute a set dense in the space is a basis if, and only if, there exists such a constant* K *that for any values* $t_1, t_2, \ldots, t_q$ *and for* $p \leqslant q$ *condition* (2) *is satisfied* (see [2]).

The existence of a basis in the space $C(D/H)$, and more generally in any space $C(Q/H)$, may be deduced, the norm of the basis being disregarded, from theorem W by the use of the following lemmas.

A. If Q is an infinite metric space and R a one-dimensional space, then the spaces $C(Q)$ and $C(Q)\times R$ are isomorphic.

B. For every compact metric space Q and for every closed subset H of that space there exists a compact metric space Q_1 such that the spaces $C(Q/H)$ and $C(Q_1)$ are isomorphic.

We shall now proceed to the proof of Theorem 1 assuming that D is the Hilbert cube (this is the only essential case). The proof is based on the following remark of Professor S. Mazur:

The sequence (e_n) *of elements of the Banach space* E *is a basis with the norm 1 if, and only if, there exists a sequence* (R_n) *of linear subsets of space* E *satisfying the following conditions:*

I. *the space* R_n *is n-dimensional,*

II. $R_n \subset R_{n+1}$ $(n=1,2,\ldots)$,

III. *the sum of linear sets* R_n *is dense in* E,

IV. *there exists a sequence of linear operations* U_n *defined on* E *and such that* $U_n(E)=R_n$, $\|U_n\|=1$, $U_n(x)=x$ for $x \in R_n$ $(n=1,2,\ldots)$.

In order to obtain the basis (e_n), the sets R_n and the operations U_n being given, it is sufficient, as can easily be verified, to take for e_1 an arbitrary element of R_1 different from 0, and for e_{n+1} $(n=1,2,\ldots)$ such an element of R_{n+1} different from 0 that $U_n(e_{n+1})=0$, and to observe that the sequence (e_n) thus defined satisfies the assumption of theorem (x). Conversely, if (e_n) is a basis with the norm 1, then for $x=\sum_{i=1}^{\infty} t_i e_i \in E$ we define $U_n(x)=\sum_{i=1}^{n} t_i e_i$ and take

$$R_n = U_n(E) \quad (n=1,2,\ldots).$$

Let $t=(\tau_n)$; let us introduce the notation

$$A_k(t) = (\tau_1, \tau_2, \ldots, \tau_k, 0, 0, \ldots), \tag{3}$$

$$D_k = A_k(D) \tag{4}$$

for $k=1,2,\ldots$, the Hilbert cube D being defined by formula (1).

Consider the sequence of triangulations T_k of k-dimensional cubes D_k satisfying the following conditions:

1° the diameters of the k-dimensional simplexes of the triangulation T_k are less than $1/3k$,

2° if Δ^{k+1} is a $k+1$-dimensional simplex of the triangulation T_{k+1}, then there exists such a k-dimensional simplex Δ^k of the triangulation T_k that

(5) $$A_k(\Delta^{k+1}) \subset \Delta^k.$$

The geometrical meaning of condition 2° is that the triangulation T_{k+1} consists of a group of triangulations of "prisms" whose bases are k-dimensional simplexes of the triangulation T_k. It follows hence in particular that these triangulations have also the following property:

3° if $t=(\tau_1,\tau_2,\ldots,\tau_k,0,\ldots)$ is a vertex of the triangulation T_k, then both t and $t'=(\tau_1,\tau_2,\ldots\tau_k,\frac{1}{k+1},0,\ldots)$ are vertices of the triangulation T_{k+1}.

The existence of triangulations with these properties is obvious.

Denote by Z_i the set of those vertices of the triangulation T_i which are vertices of simplexes containing in their interior the points of the set $A_i(H)$.

Let us arrange the vertices of these triangulations in an infinite sequence,

(6) $$t_1,t_2,\ldots,t_{k_{j_1}},\ t_{k_{j_1}+1},t_{k_{j_2}},t_{k_{j_2}+1},\ldots,\ t_{k_{j_3}},\ldots,$$

in such a way as to make the points $t_1,t_2,\ldots,t_{k_{j_i}}$ exhaust all the vertices of the triangulation T_{j_i} that do not belong to Z_{j_i}, (j_i) being an increasing sequence of the numbers of all the triangulations which contain at least one vertex that does not belong to the corresponding set Z_{j_i}.

Taking into account formula (1), property 1° and the fact that the set H is closed, it is easy to verify that

(xx) if $t \in D-H$, then there exists such a k that the point k is contained in a simplex of triangulation T_k no vertex of which belongs to Z_k.

Write

(7) $$\iota(n)=\inf_{n\leqslant j_i} j_i.$$

We shall now define the linear sets R_n. To each system k of real numbers $(a_1,a_2,\ldots,a_k)$ $(k=1,2,\ldots)$ we shall assign a function belonging to $C(D/H)$, which will be denoted by $[a_1,a_2,\ldots,a_n]$ in such a way as to satisfy the following conditions:

(a) $[a_1](t_j) \qquad =0 \quad \text{for} \quad j>1,$

(b) $[a_1,a_2,\ldots,a_n](t_j)=a_j \quad \text{for} \quad j=1,2,\ldots,n \ (n=1,2,\ldots),$

(c) $[a_1,a_2,\ldots,a_n](t_j)=[a_1,a_2,\ldots,a_{n-1}]\ (A_{\iota(n-1)}(t_j))$
for $j=n+1,\ n+2,\ldots,k_{\iota(n)}$ $(n=1,2,\ldots)$,

(d) if $t \in D_{\iota(n)}$, t having in the triangulation $T_{\iota(n)}$ a barycentric representation

$$t=\sum_{j=1}^{k_{\iota(n)}} \lambda_j t_j+\sum_j \bar{\lambda}_j v_j,$$

where $v_j \varepsilon Z_{\iota(n)}$, then

$$[a_1, a_2, \ldots, a_n](t) = \sum_{j=1}^{k_\iota(n)} \lambda_j [a_1, a_2, \ldots, a_n](t_j),$$

(e) for any $t \in D$

$$[a_1, a_2, \ldots, a_n](t) = [a_1, a_2, \ldots, a_n]\left(A_{\iota(n)}(t)\right).$$

It is easy to prove by induction that functions with the above properties exist and are uniquely defined by those properties.

From the definition of sequence (6) and from (d) it immediately follows that these functions belong to the space $C(D/H)$.

Let us take as R_n the set of functions of the form $[a_1, a_2, \ldots, a_n]$ assigned to all systems of n numbers. It is easy to show on the basis of the property (xx) and on (a)-(e) that the sets defined in this way are linear and satisfy conditions I-IV.

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MATHÉMATIQUE

Sur une généralisation d'un problème de Z. Szmydt concernant l'équation $u_{xy}=f(x,y,u,u_x,u_y)$

par

A. LASOTA

Présenté par T. WAŻEWSKI, le 26 Septembre 1956

Z. Szmydt [1], [2] a établi l'existence d'une solution du système de n équations

$$\text{(I)}\qquad u_{xy}^{(i)}(x,y)=f_i(x,y,u^{(1)},\dots,u^{(n)},u_x^{(1)},\dots,u_x^{(n)},u_y^{(1)},\dots,u_y^{(n)})\qquad (i=1,\dots,n),$$

assujettie à certaines conditions de régularité et aux conditions suivantes:

$$u_x^{(i)}(x,y)=g_i(x,u^{(1)}(x,y),\dots,u^{(n)}(x,y),u_y^{(1)}(x,y),\dots u_y^{(n)}(x,y))\quad\text{pour}\quad (x,y)\in C_1:\quad y=k(x)\quad (i=1,\dots,n),$$

$$\text{(II)}\qquad u_y^{(i)}(x,y)=h_i(y,u^{(1)}(x,y),\dots,u^{(n)}(x,y),u_x^{(1)}(x,y),\dots,u_x^{(n)}(x,y))\quad\text{pour}\quad (x,y)\in C_2:\quad x=l(y)\quad (i=1,\dots,n),$$

$$u^{(i)}(x_0,y_0)=a_i\qquad (i=1,\dots,n).$$

Ce problème peut être interprété géometriquement comme suit (en le bornant, pour simplifier, au cas d'une seule équation):

Donnons-nous, dans l'espace des 3 variables, x, y, u, 2 surfaces cylindriques engendrées par des droites parallèles à l'axe u et dont les directrices sont les courbes C_1 et C_2. Faisons correspondre à tout point de ces surfaces un cône de Monge l'ayant pour sommet. Le problème consiste à trouver une telle solution $u(x,y)$ de l'équation (I) que la surface $u=u(x,y)$ soit tangente en tout point de son intersection avec les cylindres au cône de Monge assigné à ce point et, en outre, que cette surface passe par un point donné d'avance.

T. Ważewski, qui a indiqué cette interprétation géométrique, a proposé de remplacer les cylindres par des surfaces quelconques, images géométriques des fonctions $y=k(x,u)$ et $x=l(y,u)$ respectivement. C'est cette généralisation qui est l'objet de la communication présente.

§ 1. Théorème. *Soient* $f(x,y,u,p,q)$, $g(x,y,q)$, $h(y,u,p)$, $k(x,u)$ *et* $l(y,u)$ *des fonctions continues qui, pour* $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$ *et* $u,p,q,\bar{u},\bar{p},\bar{q}$ *quelconques, satisfont aux conditions:*

(1) $$|f(x,y,u,\bar{p},\bar{q}) - f(x,y,u,p,q)| \leqslant L|\bar{p}-p| + M|\bar{q}-q|,$$

(2) $$|k(x,\bar{u}) - k(x,u)| \leqslant N|\bar{u}-u|, \qquad |l(y,\bar{u}) - l(y,u)| \leqslant N|\bar{u}-u|,$$

(3) $$|g(x,u,\bar{q}) - g(x,u,q)| \leqslant R|\bar{q}-q|, \qquad |h(y,u,\bar{p}) - h(y,u,p)| \leqslant S|\bar{p}-p|,$$

$$|f(x,y,u,p,q)| \leqslant K, \qquad |g(x,u,q)| \leqslant K, \qquad |h(y,u,p)| \leqslant K,$$

(4) $$0 \leqslant k(x,u) \leqslant b \quad \text{et} \quad 0 \leqslant l(y,u) \leqslant a,$$

dans lesquelles

(5) $$(\max(a,b)+1)KN < 1, \quad bL < 1, \quad aM < 1 \quad \textit{et} \quad \frac{RS}{(1-bL)(1-aM)} < 1.$$

Soit de plus π *le rectangle défini par les conditions*

$$0 \leqslant x \leqslant a, \qquad 0 \leqslant y \leqslant b.$$

Soient enfin u_0 *un nombre réel quelconque et* (x_0, y_0) *un point arbitraire de* π.

Cela posé, il existe une fonction $u(x,y)$ *satisfaisant aux* 5 *conditions:*

1° $u(x,y)$ *est de classe* C^1; *sa dérivée* $u_{xy}(x,y)$ *existe et elle est continue dans le rectangle* π;

2° *elle y satisfait à l'équation*

$$u_{xy} = f(x,y,u,u_x,u_y);$$

3° *on a en tout point* $P(\bar{x},\bar{y},\bar{u})$ *d'intersection des surfaces* $y=k(x,y)$ *et* $u=u(x,y)$ *l'égalité*

$$u_x(\bar{x},\bar{y}) = g(\bar{x},\bar{u},u_y(\bar{x},\bar{y}));$$

4° *on a en tout point* $Q(\bar{\bar{x}},\bar{\bar{y}},\bar{\bar{u}})$ *d'intersection des surfaces* $x=l(y,u)$ *et* $u=u(x,y)$ *l'égalité*

$$u_y(\bar{\bar{x}},\bar{\bar{y}}) = h(\bar{\bar{y}},\bar{\bar{u}},u_x(\bar{\bar{x}},\bar{\bar{y}}));$$

5° $$u(x_0,y_0) = u_0.$$

§ 2. La démonstration de ce théorème a recours au théorème bien connu de Schauder [3] sur le point invariant.

Désignons par Z l'ensemble de toutes les fonctions $u(x,y)$ de classe C^1 assujetties en tout point de π aux inégalités

(7) $$|u_0 - u(x,y)| \leqslant (ab+a+b)K, \qquad |u_x(x,y)| \leqslant (b+1)K,$$
$$|u_y(x,y)| \leqslant (a+1)K,$$

(8) $$|u_x(x,y+\varepsilon) - u_x(x,y)| \leqslant K\varepsilon, \qquad |u_y(x+\varepsilon,y) - u_y(x,y)| \leqslant K\varepsilon,$$

(9) $$|u_x(x+\varepsilon,y) - u_x(x,y)| \leqslant m(\varepsilon), \qquad |u_y(x,y+\varepsilon) - u_y(x,y)| \leqslant n(\varepsilon),$$

où $m(\varepsilon)$ et $n(\varepsilon)$ sont des fonctions positives, croissantes de $\varepsilon>0$ et telles que

$$\lim_{\varepsilon\to 0} m(\varepsilon)=\lim_{\varepsilon\to 0} n(\varepsilon)=0. \tag{10}$$

Elles seront définies dans la suite.

Introduisons la norme suivante dans l'espace E des fonctions continues dans le rectangle π avec leurs dérivées du premier ordre:

$$\sup_{\pi}|u(x,y)|+\sup_{\pi}|u_x(x,y)|+\sup_{\pi}|u_y(x,y)|.$$

Ainsi défini, E n'est un que dans cet espace linéaire normé et complet de Banach et l'on voit aisément que l'ensemble Z y est non-vide, compact et convexe. Définissons maintenant une transformation T de l'ensemble Z en faisant correspondre à toute fonction $u(x,y)$ appartenant à Z une fonction $u(x,y)$ appartenant à Z. La construction de $u(x,y)$ comprend les deux étapes suivantes:

(i) La courbe d'intersection des surfaces $y=k(x,y)$ et $u=u(x,y)$ étant donnée par les équations $y=y(x)$ et $u=u(x,y(x))$, où $y(x)$ est une solution de l'équation

$$y(x)=k\big(x,u(x,y(x))\big), \tag{11}$$

on montre aisément à l'aide des inégalités (2), (5) et (7) que l'équation (11) n'a qu'une seule solution qui soit continue dans l'intervalle fermé $0\leqslant x\leqslant a$. La démonstration s'appuie sur le principe de Banach-Tichonov-Cacciopoli.

On détermine de la même manière la courbe $x=x(y)$, $u=u(x(y),y)$ d'intersection des surfaces $x=l(y,u)$ et $u=u(x,y)$, en se servant de l'équation

$$x(y)=l\big(y,u(x(y),y)\big). \tag{12}$$

(ii) On définit la fonction $u(x,y)$ par la formule

$$\begin{aligned}\widetilde{u}(x,y)=u_0&+\int_{x_0}^{x}d\xi\int_{y(\xi)}^{y}f\big(\xi,\eta,u(\xi,\eta),u_x(\xi,\eta),u_y(\xi,\eta)\big)\,d\eta+\\&+\int_{y_0}^{y}d\eta\int_{x(\eta)}^{x_0}f\big(\xi,\eta,u(\xi,\eta),u_x(\xi,\eta),u_y(\xi,\eta)\big)\,d\xi+\\&+\int_{x_0}^{x}g\big(\xi,u(\xi,y(\xi)),u_y(\xi,y(\xi))\big)\,d\xi+\int_{y_0}^{y}h\big(\eta,u(x(\eta),\eta),u_x(x(\eta),\eta)\big)\,d\eta.\end{aligned} \tag{13}$$

On peut aisément vérifier que la transformation T ainsi définie est continue dans l'espace E. Cela résulte de la forme de (13) et des propriétés de (7) et (8). En différentiant les deux membres de (13), on conclut que toute fonction $u(x,y)$ qui constitue le point invariant de la transformation T satisfait à la thèse du théorème.

Pour achever la démonstration, il suffit donc d'établir l'existence des fonctions $m(\varepsilon)$ et $n(\varepsilon)$ telles que l'on ait $T(Z)\subset Z$. Or, vu que, en vertu de la forme de (13), T transforme les fonctions $u(x,y)$ de classe C^1 dans π assujetties aux conditions (7) et (8), en fonctions de classe C^1 qui y satisfont aux mêmes conditions, il reste à choisir les fonctions $m(\varepsilon)$ et $n(\varepsilon)$ de manière que la transformation T conserve les relations (9), c'est-à-dire que les relations (4), (8) et (9) entraînent les suivantes:

$$(9') \qquad |\widetilde{u}_x(x+\varepsilon,y)-\widetilde{u}_x(x,y)|\leqslant m(\varepsilon), \qquad |\widetilde{u}_y(x,y+\varepsilon)-\widetilde{u}_y(x,y)|\leqslant n(\varepsilon).$$

Il résulte de (10), (11) et (12) que

$$|\widetilde{u}_x(x+\varepsilon,y)-\widetilde{u}_x(x,y)|\leqslant\delta_1(\varepsilon)+bLm(\varepsilon)+Rn(\delta_2(\varepsilon))$$

et

$$|\widetilde{u}_x(x,y+\varepsilon)-\widetilde{u}_y(x,y)|\leqslant\delta_3(\varepsilon)+aMn(\varepsilon)+Sm(\delta_4(\varepsilon)),$$

où $\delta_i(\varepsilon)$ s'exprime pour $i=1,2,3$ et 4 à l'aide de modules de continuité uniforme des fonctions f,g,h,k et l envisagées pour

$$(x,y)\in\pi,\ |u-u_0|\leqslant(ab+a+b)K, \qquad |p|\leqslant(b+1)K, \qquad |q|\leqslant(a+1)K$$

de façon que $\lim_{\varepsilon\to 0}\delta_i(\varepsilon)=0$ pour $i=1,2,3$ et 4.

Il suffit par conséquent que les fonctions $m(\varepsilon)$ et $n(\varepsilon)$ satisfassent aux conditions

$$\delta_1(\varepsilon)+bLm(\varepsilon)+Rn(\delta_2(\varepsilon))=m(\varepsilon)$$

et

$$\delta_3(\varepsilon)+aMn(\varepsilon)+Sm(\delta_4(\varepsilon))=n(\varepsilon).$$

Ce système de deux équations pour $m(\varepsilon)$ et $n(\varepsilon)$ n'a pour $\varepsilon>0$ qu'une seule solution positive satisfaisant à la condition (10). On peut l'obtenir, en vertu de l'hypothèse (5), par la méthode des approximations successives.

Le théorème se trouve ainsi démontré.

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MATHEMATICS

A Fixed Point Theorem in Hilbert Space

by

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In the present note a fixed point theorem is given for a weakly continuous, generally non-linear, operator in a Hilbert space. This theorem is applicable to the theory of non-linear integral equations. For a completely continuous operator in Hilbert space a fixed point theorem with an analogous condition is given by M. A. Krasnoselsky [1], who has also given an application of his theorem to the theory of non-linear integral equations of the Hammerstein type.

1. Let $\mathfrak{H}$ be a real separable Hilbert space. Denote by Q the sphere in $\mathfrak{H}$ with centre in 0 and radius r, where 0 is the zero-element in $\mathfrak{H}$, i. e., the set of all elements of $\mathfrak{H}$ satisfying the inequality $\|x\| \leqslant r$. The boundary of Q, i. e., the set of all elements of $\mathfrak{H}$ satisfying the equation $\|x\| = r$, is denoted by S.

An operator F is said to be weakly closed if it has the following property:

The conditions: 1° x_n converges weakly to x and 2° Fx_n converges weakly to y, imply that $y = Fx$.

The following theorem is true.

THEOREM 1. *Let F be a weakly closed operator defined on Q with range in $\mathfrak{H}$. Suppose that the operator F maps the sphere Q into a bounded set in $\mathfrak{H}$ provided the following condition is fulfilled:*

$$(1) \qquad (F(x), x) \leqslant (x, x) \quad \textit{for every} \quad x \text{ of } S.$$

Under these assumptions F possesses a fixed point in Q, i. e., there exists an element x_0 in Q such that

$$F(x_0) = x_0.$$

The proof of Theorem 1 is based on the classical theorems of the combinatorial topology concerning the notion of the degree of a mapping defined on a sphere of a finite dimensional Euclidean space.

An operator F is said to be weakly continuous if it is continuous under the weak convergence of elements of the space, i. e., the condition that x_n converges weakly to x implies that $F(x_n)$ converges weakly to $F(x)$.

As an immediate corollary of Theorem 1 we obtain:

THEOREM 2. *Let F be a weakly continuous operator defined on Q with range in $\mathfrak{H}$. If condition* (1) *is satisfied then F possesses a fixed point in Q.*

Remark. The condition that $\mathfrak{H}$ is separable is not necessary for the validity of Theorems 1 and 2 and can be replaced by the weaker restriction that the range of F is a separable subset in H.

Let F be a weakly continuous operator defined on the sphere Q of a separable real Hilbert space. We suppose also that $F(x)$ is differentiable in the sense of Fréchet at the point 0. Denote by $A=F'(0)$ the Fréchet differential at the point 0.

We shall say that a linear operator A is non-positive if the following condition is satisfied:

$$(Ax,x) \leqslant 0 \quad \text{for every} \quad x \text{ of } H.$$

THEOREM 3. *Let F be a weakly continuous operator defined on the sphere Q of a real separable Hilbert space. Let us assume that the following conditions are satisfied;* 1° $F(0)=0$; 2° *At the point* 0 $F(x)$ *has a non-positive Fréchet differential $F'(0)=A$. Under these assumptions, if ε is a positive number not greater than r, and if Q_ε is the sphere with centre in* 0 *and radius ε, then the image of Q_ε under the transformation $\Phi(x)=x-F(x)$ contains the same sphere K_ε with centre in* 0, *i. e.,* $\Phi(Q)\supset K_\varepsilon$.

The proof of Theorem 3 is based on Theorem 2.

2. We shall now make some remarks concerning the application of Galerkin's method to the equation

$$F(x)=x. \tag{2}$$

It appears that the Galerkin approximations to the solution of equation (2) always exist whenever condition (1) is satisfied.

We shall give a more precise wording of this statement.

Let (e_n) $(n=1,2,\ldots)$ be an orthonormal complete system in a separable Hilbert space $\mathfrak{H}$. Denote by H_n the n-dimensional space spanned by the elements e_i $(i=1,2,\ldots,n)$. If x_n is of the form

$$x_n=\sum_{i=1}^{n} c_i^{(n)} e_i, \tag{3}$$

where the numbers $c_i^{(n)}$ are defined by the system of equations

$$(Fx_n, e_i)=c_i^{(n)} \quad (i=1,2,\ldots,n) \tag{4}$$

then x_n is a Galerkin approximation to the solution of equation (2).

THEOREM 4. *Let F be a weakly continuous operator defined on the sphere Q of a real separable Hilbert space $\mathfrak{H}$ with range in $\mathfrak{H}$. If F satisfies condition* (1), *then for every n $(n=1,2,\ldots)$ there exists a Galerkin approximation $x_n \in H_n \cdot Q$. If the x_n, where $x_n \in H_n \cdot Q$, form a sequence of Galerkin approximations then there exists a sub-sequence $(x_{\underline{n}})$ which converges weakly to a solution of equation* (2). *Every weakly convergent subsequence (x_{n_p}), $x_{n_p} \in H_{n_p} \cdot Q$ of Galerkin approximations converges weakly to a solution of equation* (2). *If equation* (2) *has only one solution in Q, then the sequence (x_n), $x_n \in H_n \cdot Q$ of Galerkin approximations converges weakly to this solution of equation* (2).

We shall now give an example of an application of Theorem 1 to the theory of non-linear integral equations. Consider the following integral equation of Hammerstein's type

$$\varphi(s)=\int_G k(s,t)f\big(t,\varphi(t)\big)\,dt, \tag{5}$$

where G is a closed bounded set of a finite dimensional Euclidean space; the function $f(t,u)$, defined for $t \in G$ and $-\infty<u<+\infty$, is continuous for a fixed t and measurable for a fixed u. The function $f(t,u)$ defines an operator F as follows:

$$y=Fx, \quad \text{where} \quad y(t)=f\big(t,x(t)\big), \quad t \in G. \tag{6}$$

Denote by $L_2(G)$ the real Hilbert space of all square integrable functions in G. Let us assume that the kernel $k(s,t)$ $(s,t \in G)$ defines a linear bounded operator K with domain and range in $L_2(G)$:

$$y=Kx, \quad \text{where} \quad y(s)=\int_G k(s,t)x(t)\,dt, \quad x,y \in L_2(G). \tag{7}$$

Concerning the operator K we suppose that the following decomposition holds:

$$K=HH^*, \tag{8}$$

where H is a linear bounded operator defined on $L_2(G)$ with range in $L_p(G)$, $p>1$, and H^* denotes the operator conjugate to H. Let us assume that operator H is defined by the kernel $h(s,t)$ and the equation

$$y(s)=\int_G h(s,t)x(t)\,dt, \quad \text{where} \quad y=Hx \ (s,t \in G) \tag{9}$$

and $x \in L_2(G)$, $y \in L_p(G)$. Suppose also that

$$\int_G h^2(s,t)\,dt<\infty \tag{10}$$

for almost every fixed $s \in G$.

THEOREM 5. *Let us assume that the operator* (6) *is defined on* $L_p(G)$ *with values in* $L_q(G)$, *where* $p^{-1}+q^{-1}=1$. *We require of the function* $f(t,u)$ *that the following condition be fulfilled*:

$$f(t,u)u \leqslant au^2+b(t)u^{2-\gamma}+c(t),$$

where $0<\gamma<2$, $b(t)\in L_{2/\gamma}(G)$, $c(t)\in L(G)$ *and for some* $r>0$

$$a\|K\|r^2+\Big[\int_G |b(t)|^{2/\gamma}dt\Big]^{\gamma/2}\|K\|^{1-\gamma/2}r^{2-\gamma}+\int_G c(t)\,dt<r^2.$$

Suppose also that operator (7) *satisfies condition* (8), *and operator* (9) *satisfies condition* (10). *Under these assumptions there exists in* $L_2(G)$ *at least one solution of equation* (5).

The proof of Theorem 5 makes use of some results concerning the operator (6), which were obtained by Krasnoselsky [2].

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MATHEMATICS

On Ritz's Method

by

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1. This note contains some remarks concerning the well-known Ritz method and the method of least squares. In our considerations we do not use the variation method. The argument given is of an exclusively geometrical character and is based on the notion of orthogonal projection and on the orthogonalization process. By an iteration method given below we can obtain the approximate solutions without solving the corresponding systems of ordinary linear equations.

Let A be a linear (i. e., additive and homogeneous) operator with domain $D(A)$ and range in a separable Hilbert space H. Suppose that $D(A)$ is a dense linear set in H. Consider the linear equation

$$Au=f \qquad u,f\in H, \tag{1}$$

where A is a linear self-adjoint and positively defined operator, i. e., $A=A^*$, where A^* denotes the adjoint of A, and there exists a positive number k such that

$$(Au,u)\geqslant k\|u\|^2. \tag{2}$$

We now use the construction of an auxiliary Hilbert space H_A, due to K. Friedrichs [1]. The scalar product in H_A is $[u,v]=(Au,v)$, H_A is the completion of $D(A)$ in the norm $|u|=(Au,u)^{1/2}$. H_A is separable. Condition (2) implies

$$|u|^2\geqslant k\|u\|^2, \tag{3}$$

and

$$\|Au\|\geqslant k\|u\|. \tag{4}$$

It follows by (3) that the linear functional in $H\,(u,f)$ is also a linear functional in H_A. Hence there exists an element f' in H_A such that $(u,f)=$ $=[u,f']$. But $[u,f']=(Au,f')$ and we infer that $f=A^*f'$. Since A is self-adjoint we obtain $f=Af'$, i. e., $x=f'$ is the unique solution of equation (1).

In order to obtain the approximate solution by Ritz's method we should take a complete system (φ_i) in the space H_A. Suppose also that the elements φ_i are linearly independent. Denote by H_n the linear space spanned by the elements $\varphi_1, \varphi_2, \ldots, \varphi_n$ and by P_n the orthogonal projection on H_n in the space H_A. Then $x_n = P_n x$ is Ritz's approximate solution. In fact, let us assume that $x_n = \sum_{k=1}^{n} a_k \varphi_k$. Since $x - P_n x$ and φ_i are orthogonal for $i = 1, 2, \ldots, n$, $\left[x - \sum_{k=1}^{n} a_k \varphi_k, \varphi\right] = 0$, hence $\sum_{k=1}^{n} a_k (A\varphi_k, \varphi_i) = (f, \varphi_i)$, $i = 1, 2, \ldots, n$, and we obtain the well-known Ritz equations.

2. The application of Ritz method in practice involves great difficulties in finding a complete system (φ_n) in the space H_A.

A sufficient condition for (φ_n) to be complete in H_A is that the system $(A\varphi_n)$ be complete in H [2]. But under this assumption the following method can be used, which coincides in its principle but not in form with the well-known method of least squares. Let us assume that the linear operator A is self-adjoint and positively defined on a linear set $D(A)$ being dense in H. Let (φ_n) be a sequence of linearly independent elements from $D(A)$ such that $(A\varphi_n)$ is complete in H. Denote by $A(H_n)$ the space spanned by the elements $A\varphi_1, A\varphi_2, \ldots, A\varphi_n$. Denote by P_n the orthogonal projection on $A(H_n)$ in the space H. Let $x_n = \sum_{k=1}^{n} a_k \varphi_k$ be an element of H_n such that $Ax_n = P_n f$. Then x_n is the approximate solution, and x_n converges to a solution of equation (1). In fact, Ax_n converges to f, and x_n is convergent by (4). Since the operator A is closed, we have $x_n \to x$ and $Ax = f$.

Observe that $f - P_n f$ and $A\varphi_i$ are orthogonal, i. e., $(f - P_n f, A\varphi_i) = 0$ for $i = 1, 2, \ldots, n$. Hence we have

$$\sum_{k=1}^{n} a_k (A\varphi_k, A\varphi_i) = (f, A\varphi_i) \qquad (i = 1, 2, \ldots, n), \tag{*}$$

i. e., we obtain the well-known equations for the method of least squares.

We can also obtain the approximate solutions x_n in the following way.

We construct an orthonormal system (e_n) in $A(H_n)$. $e_1 = \dfrac{A\varphi_1}{\|A\varphi_1\|} = A\widetilde{\varphi}_1$, where $\widetilde{\varphi}_1 = \dfrac{\varphi_1}{\|A\varphi_1\|}$. If the $e_i = A\widetilde{\varphi}_i$ are already defined for $i = 1, 2, \ldots, n$, then we put $e_{n+1}' = A\varphi_{n+1} - \sum_{i=1}^{n} (A\varphi_{n+1} A\widetilde{\varphi}_i) A\widetilde{\varphi}_i$ and $e_{n+1} = \dfrac{e_{n+1}'}{\|e_{n+1}'\|} = A\widetilde{\varphi}_{n+1}$, where

$$\widetilde{\varphi}_{n+1} = \varphi_{n+1} - \sum_{i-1}^{n} (A\varphi_{n+1}, A\widetilde{\varphi}_i) \widetilde{\varphi}_i. \tag{5}$$

The sequence $(\widetilde{\varphi}_n)$ is defined by iteration process (5). Hence $P_n f = = \sum_{i=1}^{n} (f, e_i) e_i = \sum_{i=1}^{n} (f, A\widetilde{\varphi}_i) A\widetilde{\varphi}_i = A_{x_n}$, where $x_n = \sum_{i=1}^{n} (f, A\widetilde{\varphi}_i)\widetilde{\varphi}_i$. If $x_1 = (f, e_1)\widetilde{\varphi}_1$, then $Ax_1 = P_1 f$. Putting

$$x_{n+1} = x_n + (f, A\widetilde{\varphi}_{n+1})\widetilde{\varphi}_{n+1}, \tag{6}$$

we obtain an iteration process for the approximate solution of equations (1) defined by formulas (5) and (6).

The following modification of the iteration process (5)-(6) is more convenient for computation since it contains no square roots connected with definition of the norm in a Hilbert space.

We set

$$\bar{\varphi}_1 = \varphi_1, \ e_1 = \frac{A\bar{\varphi}_1}{\|A\bar{\varphi}_1\|}, \qquad x_1 = \frac{(f, A\bar{\varphi}_1)}{\|A\varphi_1\|^2}\bar{\varphi}_1,$$

$$\bar{\varphi}_{n+1} = \varphi_{n+1} - \sum_{i=1}^{n} \frac{1}{\|A\bar{\varphi}_i\|^2}(A\varphi_{n+1}, A\bar{\varphi}_i)\bar{\varphi}_i. \tag{7}$$

Then

$$x_{n+1} = x_n + \frac{(f, A\bar{\varphi}_{n+1})}{\|A\bar{\varphi}_{n+1}\|^2}\bar{\varphi}_{n+1}. \tag{8}$$

It is easy to see that

$$e_n = \frac{A\bar{\varphi}_n}{\|A\bar{\varphi}_n\|} \qquad (n = 1, 2, \dots)$$

form an orthonormal system in H and that $P_n f = Ax_n$ for $n = 1, 2, \dots$

An estimate of error can be given as follows:

$$\|f - Ax_n\|^2 = \|f - P_n f\|^2 = \|f\|^2 - \|P_n f\|^2 = \|f\|^2 - \|Ax_n\|^2.$$

By (8) we have

$$\|P_n f\|^2 = \sum_{k=1}^{n} \frac{|(f, A\bar{\varphi}_k)|^2}{\|A\bar{\varphi}_k\|^2}.$$

From (4) we have

$$\|x_n - x\|^2 \leqslant \frac{1}{k}\left[\|f\|^2 - \sum_{k=1}^{n} \frac{|(f, A\bar{\varphi}_k)|^2}{\|A\bar{\varphi}_k\|^2}\right] = \frac{1}{k}\delta_n.$$

For the approximate solution x_{n+1} we obtain the following error estimate:

$$\|x_{n+1} - x\|^2 \leqslant \frac{1}{k}\left[\delta_n - \frac{|(f, A\bar{\varphi}_{n+1})|^2}{\|A\bar{\varphi}_{n+1}\|^2}\right].$$

Thus we obtain a recurrent formula for the estimate of error:

$$(9)\qquad \delta_1=\|f\|^2-\frac{|(f,A\varphi_1)|^2}{\|A\varphi_1\|^2},\qquad \delta_{n+1}=\delta_n-\frac{|(f,A\overline{\varphi}_{n+1})|^2}{\|A\overline{\varphi}_{n+1}\|^2},$$

where $\delta_n=\|f-Ax_n\|^2$. It is evident that $\delta_n=\|f-P_nf\|^2=\min\limits_{g\in A(H_n)}\|f-g\|$.

It is easy to see that the sequence of approximate solutions given by formula (6) converges to the solution of linear equation (1) if the following conditions are satisfied:

1° the linear operator A satisfies condition (2);

2° there exists a solution of linear equation (1);

3° the closed linear subspace of H spanned by the elements $A\varphi_1$, $A\varphi_2,\ldots$ contains the element f.

In this case we have $\delta_n=\|f-Ax_n\|\to 0$ as $n\to\infty$.

The Ritz's method and the method of least squares are closely connected [2]. We can easily show this in a geometrical way as well. If A is a linear self-adjoint, positively defined operator, then there exists a self-adjoint, positively defined operator B such that $B^2=A$.

If (φ_i) is a linearly independent complete system in H_A, then $(B\varphi_i)$ is a complete system in H.

For if $(B\varphi_i,f)=0$ for $i=1,2,\ldots$, then setting $f=Bg$, where $g\in H_A$, we have $(B\varphi_i,f)=(B\varphi_i,Bg)=(A\varphi_i,g)=[\varphi_i,g]=0$. Hence $g=0$, and consequently $f=0$. Thus we can apply the method of least squares to equation $Bu=B^{-1}f$, which is equivalent to equation (1). Then we should write equations (*) for this case, i. e., $\sum\limits_{k=1}^{n}a_k(B\varphi_k,B\varphi_i)=(B^{-1}f_1B\varphi_i)$ $(i=1,2,\ldots,n)$. But $(B\varphi_k,B\varphi_i)=(A\varphi_k,\varphi_i)$ and $(B^{-1}f,B\varphi_i)=(f,\varphi_i)$ and the Ritz's equations are obtained.

3. S. G. Michlin [3] has recently proved the following theorem:

Let us assume that A,B are self-adjoint and positively defined linear operators such that

$$D(A)=D(B)\quad\text{and}\quad D(\sqrt{A})=D(\sqrt{B}).$$

If the spectrum of B is discrete and (φ_n) $n=1,2,\ldots$ is a complete orthonormal system of its eigen vectors, then $Au_n\to f$ as $n\to\infty$, where $u_n=\sum\limits_{i=1}^{n}a_i\varphi_i$ is the approximate solution of (1) by Ritz's method.

It is easy to see that a similar but more general theorem holds for the method of least squares or for the method given in **2.**:

If $D(A)=D(B)$ and (φ_n) $n=1,2,\ldots$ is a linearly independent system such that $(B\varphi_n)$ is complete in H, then the method of least squares or the method given in **2.** are applicable to the system of co-ordinates (φ_n).

Hence $Au_n \to f$, as $n \to \infty$, where u_n is the corresponding approximate solution.

In fact, in virtue of **2.**, it is sufficient to show that $(A\varphi_n)$ forms a complete system in H.

It is easy to see that the linear operator AB^{-1} is bounded. Let y be an arbitrary element of H and let v_n be a sequence of linear combinations of elements of φ_k such that $Bv_n \to BA^{-1}y$, as $n \to \infty$. Then $Av_n = AB^{-1}Bv_n \to AB^{-1}BA^{-1}y = y$, as $n \to \infty$.

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MATHEMATICS

A Simple Practical Method and a Compact Computing Scheme for the Solution of Linear Equations in Hilbert Space

by

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Presented by W. ORLICZ on October 4, 1956

1. In [1] an iteration method is given for finding the approximate solutions of a linear equation in a separable Hilbert space H

$$Au=f, \quad (u, f \in H), \tag{1}$$

where A is linear, i. e., additive and homogeneous.

In the present paper we give a compact computation scheme based on the above-mentioned method. This scheme is very convenient in practice and yields economy in computation. It is shown in [1] that the method in question is obtained in a geometrical way without solving the corresponding systems of ordinary linear equations. It appears that in the finite-dimensional case this computation scheme is the same as that of Banachiewicz applied to the solution of systems of linear algebraic equations. Thus, we obtain a geometrical basis for the algebraic method of Banachiewicz. The computation scheme in question is common to both methods: to Ritz's method and to the method of least squares, but the geometrical sense in these cases is different. If we wished to find the approximate solutions by the algebraic method, we would lose not only the geometrical sense but also the economy in computation, especially in the case of the method of least squares. The economy in computation in the case of the method of least squares consists in the following:

We need not find all the approximate solutions in consecutive order until we obtain a suitable one. But in consecutive order we find the error estimates for the corresponding approximate solutions by a recurrent formula until we obtain a suitable one. Only then do we find the corresponding approximate solution.

2. Let (φ_n) $n=1,2,\dots$ be a linearly independent system in the Hilbert space H. We shall orthogonalise this system and denote by $(\bar{\varphi}_n)$ the system obtained by an orthogonalisation process. We arrive at the system $(\bar{\varphi}_n)$ as follows:

$$\bar{\varphi}_1=\varphi_1,\ \bar{\varphi}_{k+1}=\varphi_{k+1}-\sum_{i=1}^{k}\frac{1}{\|\bar{\varphi}_i\|^2}(\varphi_{k+1},\bar{\varphi}_i)\bar{\varphi}_i. \tag{2}$$

We have the following transformation of the co-ordinate system

$$\varphi_1=\bar{\varphi}_1,\qquad \varphi_{k+1}=\sum_{i=1}^{k}\frac{1}{\|\bar{\varphi}_i\|^2}(\varphi_{k+1},\bar{\varphi}_i)\bar{\varphi}_i+\bar{\varphi}_{k+1}\qquad (k=1,2,\dots)$$

or

$$\begin{aligned}&\varphi_1=\bar{\varphi}_1,\\&\varphi_2=\gamma_{21}\bar{\varphi}_1+\bar{\varphi}_2,\\&\varphi_3=\gamma_{31}\bar{\varphi}_1+\gamma_{32}\bar{\varphi}_2+\bar{\varphi}_3,\\&\dots\dots\dots\dots\end{aligned} \tag{3}$$

We shall compute the elements of the triangle matrix (γ_{ki}). Denote by $a_{ki}=(\varphi_k,\varphi_i)$ the elements of the Gram matrix for the vector system (φ_n) $n=1,2,\dots$

Denote by β_{ki} the coefficients $(\varphi_k,\bar{\varphi}_i)$ for $i<k$ and by β_{ii} the numbers $\|\bar{\varphi}_i\|^2$. Put

$$\beta_{k1}=a_{k1}\quad \text{for}\quad k=1,2,\dots\quad \text{and}\quad \gamma_{ki}=\beta_{ki}/\beta_{ii}\quad \text{for}\quad i<k.$$

Then we obtain

$$\beta_{kj}=(\varphi_k,\bar{\varphi}_j)=\left(\varphi_k,\varphi_j-\sum_{i=1}^{j-1}\frac{1}{\|\bar{\varphi}_i\|^2}(\varphi_j,\bar{\varphi}_i)\bar{\varphi}_i\right)=(\varphi_k,\varphi_j)-\sum_{i=1}^{j-1}\frac{1}{\|\bar{\varphi}_i\|^2}\overline{(\varphi_j,\bar{\varphi}_i)}(\varphi_k,\bar{\varphi}_i).$$

Thus we arrive at the following recurrent formula:

$$\beta_{kj}=a_{kj}-\sum_{i=1}^{j-1}\bar{\gamma}_{ji}\beta_{ki}=a_{kj}-\sum_{i=1}^{j-1}\bar{\beta}_{ji}\gamma_{ki},\quad \text{for}\quad j<k, \tag{4}$$

where $\bar{a}$ denotes the complex number conjugate to a. We can obtain an analogous recurrent formula for the numbers β_{kk}. Since $(\bar{\varphi}_j,\bar{\varphi}_i)=0$ for $i\neq j$, we have by (2)

$$\beta_{kk}=(\bar{\varphi}_k,\bar{\varphi}_k)=\left(\varphi_k-\sum_{i=1}^{k-1}\frac{1}{\|\bar{\varphi}\|^2}(\varphi_k,\bar{\varphi}_i)\bar{\varphi}_i,\varphi_k-\sum_{j=1}^{k-1}\frac{1}{\|\bar{\varphi}_i\|^2}(\varphi_k,\bar{\varphi}_j)\bar{\varphi}_j\right)=$$

$$=(\varphi_k,\varphi_k)-\sum_{i=1}^{k-1}\frac{1}{\|\bar{\varphi}_i\|^2}\overline{(\varphi_k,\bar{\varphi}_i)}(\varphi_k,\bar{\varphi}_i).$$

Hence,

$$\beta_{kk}=a_{kk}-\sum_{i=1}^{k-1}\bar{\gamma}_{ki}\beta_{ki}=a_{kk}-\sum_{i=1}^{k-1}\gamma_{ki}\bar{\beta}_{ki}=a_{kk}-\sum_{i=1}^{k-1}|\beta_{ki}|^2/\beta_{ii} \tag{5}$$

Now suppose that the orthogonal system $(\bar{\varphi}_k)$ is also normalised to form the system $(\tilde{\varphi}_k)$ and that the triangle matrix (s_{ki}) corresponds to the transformation of the system (φ_k) to the orthonormal system $(\tilde{\varphi}_k)$. Then the transformation (3) should be written as follows:

$$\begin{aligned}
&\varphi_1=\|\bar{\varphi}_1\|\tilde{\varphi}_1,\\
&\varphi_2=\gamma_{21}\|\bar{\varphi}_1\|\tilde{\varphi}_1+\|\bar{\varphi}_2\|\tilde{\varphi}_2,\\
&\varphi_3=\gamma_{31}\|\bar{\varphi}_1\|\tilde{\varphi}_1+\gamma_{32}\|\bar{\varphi}_2\|\tilde{\varphi}_2+\|\bar{\varphi}_3\|\tilde{\varphi}_3,\\
&\dots\dots\dots\dots\dots\dots\dots
\end{aligned} \tag{6}$$

where

$$\tilde{\varphi}_i=\bar{\varphi}_i/\|\bar{\varphi}_i\| \qquad (i=1,2,\dots).$$

Setting $s_{jj}=\beta_{jj}^{1/2}$ and $s_{ki}=\gamma_{ki}\beta_{ii}^{1/2}=\beta_{ki}/s_{ii}$, we have by (4) and (5)

$$s_{kj}=\frac{a_{kj}-\sum_{i=1}^{j-1}\bar{s}_{ji}s_{ki}}{s_{jj}}, \qquad \text{and} \qquad s_{jj}=\Big(a_{jj}-\sum_{i=1}^{j-1}|s_{ji}|^2\Big)^{1/2}. \tag{7}$$

But formulas (7) are the same as in Banachiewicz's method of square roots [2].

Finally we conclude that the triangle matrix corresponding to the transformation of a linearly independent system (φ_k) into the orthonormal system $(\tilde{\varphi}_k)$ is the matrix obtained by splitting the Gram matrix $\|(\varphi_k,\varphi_i)\|$ of this system by Banachiewicz's method of square roots. The Banachiewicz method of splitting this Gram matrix into triangle matrices without square roots corresponds to a transformation of the linearly independent system to an orthogonal one; this follows from recurrent formulas (4) and (5).

The latter method is more convenient for computation and will be used here.

3. We shall now find the suitable approximate solution of linear equation (1) by the method of least squares given in [1], provided that A is one-to-one. Using formula (7) of [1] we obtain

$$\varphi_1=\bar{\varphi}_1, \qquad \varphi_{k+1}=\bar{\varphi}_{k+1}+\sum_{i=1}^{k}\frac{1}{\|A\bar{\varphi}_i\|^2}(A\varphi_{k+1},A\bar{\varphi}_i)\bar{\varphi}_i \tag{8}$$

or

$$\varphi_k=\sum_{i=1}^{k-1}\gamma_{ki}\bar{\varphi}_i+\bar{\varphi}_k \quad \text{for} \quad k=1,2,\dots \tag{9}$$

Hence,

$$A\varphi_k=\sum_{i=1}^{k-1}\gamma_{ki}A\bar{\varphi}_i+A\bar{\varphi}_k \quad \text{for} \quad k=1,2,\dots \tag{10}$$

Since the sequence $(A\overline{\varphi}_k)$ forms an orthogonal system, we are in a position to apply the method of **2.** to the Gram matrix (a_{ki}) of the system $(A\varphi_k)$, where

$$(11) \qquad a_{ki} = (A\varphi_k, A\varphi_i).$$

In consecutive order along the matrix row we compute the numbers $\beta_{ki} = (A\varphi_k, A\overline{\varphi}_i)$ and γ_{ki}, $i=1,2,\ldots,k-1$ as in the scheme for every k. At the same time we find the numbers $b_k = (f, A\overline{\varphi}_k)$ by the following formula as a consequence of (10):

$$(12) \qquad b_1 = (f, A\overline{\varphi}_1) = (f, A\varphi_1); \qquad b_k = (f, A\overline{\varphi}_k) = (f, A\varphi_k) - \sum_{i=1}^{k-1} \overline{\gamma}_{ki} b_i.$$

At the same time we also find the numbers $\delta_k = \|f - Ax_k\|^2$ by the recurrent formula ([1], formula (9))

$$(13) \qquad \delta_1 = \|f\|^2 - \frac{|b_1|^2}{\beta_{11}}, \quad \delta_k = \delta_{k-1} - \frac{|b_k|^2}{\beta_{kk}},$$

where β_{kk} is defined by (5).

Let us assume that for $k=n$ we have obtained a sufficiently small number for δ_n. Then we find the corresponding approximate solution x_n as follows. Write x_n in the form $x_n = \sum\limits_{k=1}^{n} a_k \varphi_k$. Multiplying (9) by a_k and summing from $k=1$ to $k=n$ we have

$$x_n = \sum_{k=1}^{n} a_k \varphi_k = \sum_{k=1}^{n} \Big[\sum_{i=1}^{k-1} \gamma_{ki} a_k \overline{\varphi}_i + a_k \overline{\varphi}_k\Big] = \sum_{k=1}^{n} \sum_{i=1}^{k} \gamma_{ki} a_k \overline{\varphi}_i =$$
$$= \sum_{i=1}^{k} \Big(\sum_{k=1}^{n} \gamma_{ki} a_k\Big) \overline{\varphi}_i = \sum_{i=1}^{n} \Big(\sum_{k=1}^{n} \gamma_{ki} a_k\Big) \overline{\varphi}_i.$$

Hence,

$$(14) \qquad x_n = \sum_{i=1}^{n} \Big(\sum_{k=1}^{n} \gamma_{ki} a_k\Big) \overline{\varphi}_i,$$

since $\gamma_{kk} = 1$ for $k=1,2,\ldots,n$ and $\gamma_{ki} = 0$ for $i>k$. On the other hand ([1], formula (8)) we have

$$(15) \qquad x_n = \sum_{i=1}^{n} \frac{1}{\|A\overline{\varphi}_i\|^2} (f, A\overline{\varphi}_i) \overline{\varphi}_i = \sum_{i=1}^{n} \frac{b_i}{\beta_{ii}} \overline{\varphi}_i.$$

Comparing the coefficients in (14) and (15) we obtain

$$(16) \qquad \sum_{k=1}^{n} \gamma_{ki} a_k = \frac{b_i}{\beta_{ii}} \quad \text{or} \quad \sum_{k=i}^{n} \gamma_{ki} a_k = \frac{b_i}{\beta_{ii}} \quad \text{for} \quad i=1,2,\ldots,n.$$

A computing scheme for the approximate solution of th

3 $\delta_1=\|f\|^2-\frac{1}{\beta_{11}}\|b_1\|^2$	2 $b_1=l_1$	1 $\boxed{\beta_{11}=a_{11}}$		
7 $\delta_2=\delta_1-\frac{1}{\beta_{22}}\|b_2\|^2$	6 $b_2=l_2-\bar{\gamma}_{21}b_1$	4 $\beta_{21}=a_{21}$ $\gamma_{21}=\frac{1}{\beta_{11}}\beta_{21}$	5 $\boxed{\beta_{22}=a_{22}-\bar{\gamma}_{21}\beta_{21}}$	
12 $\delta_3=\delta_1-\frac{1}{\beta_{33}}\|b_3\|^2$	11 $b_3=l_3-\sum_{i=1}^{2}\bar{\gamma}_{3i}b_i$	8 $\beta_{31}=a_{31}$ $\gamma_{31}=\frac{1}{\beta_{11}}\beta_{31}$	9 $\beta_{32}=a_{32}-\bar{\gamma}_{21}\beta_{31}$ $\gamma_{32}=\frac{1}{\beta_{22}}\beta_{32}$	10 $\beta_{33}=a_{33}-\bar{\gamma}$
18 $\delta_4=\delta_3-\frac{1}{\beta_{44}}\|b_4\|^2$	17 $b_4=l_4-\sum_{i=1}^{3}\bar{\gamma}_{4i}b_i$	13 $\beta_{41}=a_{41}$ $\gamma_{41}=\frac{1}{\beta_{11}}\beta_{41}$	14 $\beta_{42}=a_{42}-\bar{\gamma}_{21}\beta_{41}$ $\gamma_{42}=\frac{1}{\beta_{22}}\beta_{42}$	15 $\beta_{43}=a_{43}-\bar{\gamma}$ $\gamma_{43}=\frac{1}{\beta_{33}}\beta_{43}$
. . . .	. . . .	. . . .	. . . .	. . . .
$\delta_k=\delta_{k-1}-\frac{1}{\beta_{kk}}\|b_k\|^2$	$b_k=l_k-\sum_{i=1}^{k-1}\bar{\gamma}_{ki}b_i$	$\beta_{k1}=a_{k1}$ $\gamma_{k1}=\frac{1}{\beta_{11}}\beta_{k1}$	$\beta_{k2}=a_{k2}-\bar{\gamma}_{21}\beta_{k1}$ $\gamma_{k2}=\frac{1}{\beta_{22}}\beta_{k2}$	$\beta_{k3}=a_{k3}-\bar{\gamma}$ $\gamma_{k3}=\frac{1}{\beta_{33}}\beta_{k3}$
. . . .	. . . .	. . . .	. . . .	. . . .
$\delta_n=\delta_{n-1}-\frac{1}{\beta_{nn}}\|b_n\|^2$	$b_n=l_n-\sum_{i=1}^{n-1}\bar{\gamma}_{ni}b_i$	$\beta_{n1}=a_{n1}$ $\gamma_{n1}=\frac{1}{\beta_{11}}\beta_{n1}$	$\beta_{n2}=a_{n2}-\bar{\gamma}_{21}\beta_{n1}$ $\gamma_{n2}=\frac{1}{\beta_{22}}\beta_{n2}$	$\beta_{n3}=a_{n3}-\bar{\gamma}$ $\gamma_{n3}=\frac{1}{\beta_{33}}\beta_{n3}$

Notations: $a_{ik}=(A\varphi_i, A\varphi_k)$, where φ_i $(i-1,2,\ldots)$ are linearly independent elements of H, l_i

The approximate solution corresponding to δ_n is

$$x_n=\sum_{k=1}^{n}a_k\varphi_k, \quad \text{where} \quad a_n=\frac{b_n}{\beta_{nn}},\ a_i=\frac{\beta_i}{\beta_{ii}}-$$

$\bar{\gamma}$ is the complex number conjugated to γ.

The small numbers in the corners indicate the order in which the computations should be c

linear equation $Ax=f$ $(x, \in H)$ in the Hilbert space H

$_{31}\beta_{31}-\bar{\gamma}_{32}\beta_{32}$					
$_{31}\beta_{41}-\bar{\gamma}_{32}\beta_{42}$	16 $\beta_{44}=\alpha_{44}-\bar{\gamma}_{41}\beta_{41}-\bar{\gamma}_{42}\beta_{42}-\bar{\gamma}_{43}\beta_{43}$				
. . . . .	. . . .	. . .			
$_{31}\beta_{k1}-\bar{\gamma}_{32}\beta_{k2}$	$\beta_{k4}=\alpha_{k4}-\bar{\gamma}_{41}\beta_{k1}-\bar{\gamma}_{42}\beta_{k2}-\bar{\gamma}_{43}\beta_{k3}$ $\gamma_{k4}=\frac{1}{\beta_{44}}\beta_{k4}$	. . . . . .	$\beta_{kk}=\alpha_{kk}-\sum_{i=1}^{k-1}\bar{\gamma}_{ki}\beta_{ki}$		
. . . . .	. . . .	. . .	. . . .	. . .	
$_{1}\beta_{n1}-\bar{\gamma}_{32}\beta_{n2}$	$\beta_{n4}=\alpha_{n4}-\bar{\gamma}_{41}\beta_{n1}-\bar{\gamma}_{42}\beta_{n2}-\bar{\gamma}_{43}\beta_{n3}$ $\gamma_{n4}=\frac{1}{\beta_{44}}\beta_{n4}$	. . . . . .	$\beta_{nk}=\alpha_{nk}-\sum_{i=1}^{k-1}\bar{\gamma}_{ki}\beta_{ni}$ $\gamma_{nk}=\frac{1}{\beta_{kk}}\beta_{nk}$	. . . . . .	$\beta_{nn}=\alpha_{nn}-\sum_{i=1}^{n-1}\bar{\gamma}_{ni}$

$=(f, A\varphi_i)$.

$\sum_{k=i+1}^{n}\gamma_{ki}a_k$ for $i=n-1, n-2, \ldots, 2, 1.$

rried out.

Setting $i=n$ in (16) we have

$$a_n = \frac{b_n}{\beta_{nn}}. \tag{17}$$

It follows by (16) that

$$a_i = \frac{b_i}{\beta_{ii}} - \sum_{k=i+1}^{n} \gamma_{ki} a_k \quad \text{for} \quad i=n-1,\ n-2,\ldots,2,1, \tag{18}$$

and the suitable approximate solution x_n is found.

Remark 1. $\delta_n = \|f - Ax_n\|^2 \to 0$ if, and only if, the system (φ_n) $n=1,2,\ldots$ is so chosen that f can be approximated in H by linear combinations of elements $(A\varphi_n)$, $n=1,2,\ldots$ But in practice there are in general great difficulties in examining this condition. We therefore take as φ_n an arbitrary function from the domain of A satisfying the conditions of our problem (for instance, the conditions of the Dirichlet problem). Further, for this φ_n we write the n-th row of the (γ_{ki})-matrix and compute δ_n. If δ_n is too great then we must take the next function φ_{n+1}. The functions (φ_i) $i=1,2,\ldots$ should be linearly independent and the linear transformation should be one-to-one. If δ_n is sufficiently small then we find the corresponding approximate solution x_n.

Remark 2. We can easily estimate the computing economy obtained by the above method. The solution of the triangle system (16) requires $\frac{1}{2}m(m-1)$ multiplications and the same number of additions. Thus we have an economy of $\sum_{m=1}^{n-1} \frac{1}{2}m(m-1) = \frac{1}{6}(n^3 - 3n^2 + 2n - 3)$ multiplications and of the same number of additions.

4. For the solution of equation (1) by Ritz's method we should put in the reckoning scheme $a_{ki} = (A\varphi_k, \varphi_i)$, $\beta_{ki} = (A\varphi_k, \bar{\varphi}_i)$, and in formula (12) we write (f, φ_i) instead of $(f, A\varphi_i)$ and $b_k = (f, \bar{\varphi}_k)$. Formulas (17) and (18) remain the same. It is known that in this case there is in general no error estimation. Notice that system $(\bar{\varphi}_i)$ defined by (3) forms an orthogonal system in the topology induced by the scalar product $[u,v] = (Au, v)$ provided that the operator A is self-adjoint and positively defined.

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MATHEMATICS

Application of Games to Some Problems of Mathematical Logic

by

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Presented by A. MOSTOWSKI on October 11, 1956

This note contains a necessary and sufficient condition for elementary indiscernibility of models, a sufficient condition for the indiscernibility of models by means of finite sets, and some theorems obtained by these conditions. These results will be published with proofs in Fundamenta Mathemathicae. All notations concerning theories and models are those used in [1].

1. Let us consider two models M' and M'' for an elementary theory T which has no terms and a finite number of predicates only. The set of these predicates we denote by P.

DEFINITION. *The models M' and M'' are elementarily discernible if, and only if, there is a closed formula α in T such that*

$$\mathrm{stsf}_{M'}\,\alpha \quad \textit{and} \quad \mathrm{stsf}_{M''}\sim\alpha.$$

We now define the sequence of games $G_n(M', M'')$ [2]. There are two players, I and II. The move of player I is to point out an element x in one of the models (i. e., in the set-theoretical sum $|M'|+|M''|$, when $|M|$ denotes the set of individuals of M).

The move of player II is to point out in another model an element y, which is put into correspondence with the element pointed out by player I,

$$y \longleftrightarrow x.$$

If, for example, player I has pointed out an element x in model M', then player II has to point out an element in model M''. In $G_n(M', M'')$ both players make n moves. When the play is over we have n pairs of individuals $a_1'\longleftrightarrow a_1''$, $a_2'\longleftrightarrow a_2''$, ..., $a_n'\longleftrightarrow a_n''$, where $a_i' \in |M'|$, $a_i'' \in |M''|$. (Player I may point out elements in both models. For example in the above play he could point out $a_1', a_2'', a_3', \ldots$ and so on).

Player II wins if for any predicate $a \in P$ and any finite sequence $k_1, \ldots, k_{a(a)}$, where $1 \leqslant k_i \leqslant n$

$$\mathrm{stsf}_{M'}\, a(a'_{k_1}, \ldots, a'_{k_{a(a)}})$$

if, and only if,

$$\mathrm{stsf}_{M''}\, a(a''_{k_1}, \ldots, a''_{k_{a(a)}}).$$

THEOREM 1*). *Player II has a winning method for every n in the game $G_n(M', M'')$ if, and only if, M' and M'' are indiscernible.*

2. We now consider the theory T^* obtained from T by adjunction of new variables $\mathfrak{x}_1, \ldots, \mathfrak{x}_n, \ldots$ and of a binary predicate ϵ. As models of T^* we admit only models in which ϵ is interpreted as the set theoretical relation "belongs to", and the $\mathfrak{x}_i$ are interpreted as hereditarily finite sets, i. e., as finite sets whose elements are individuals, finite sets of individuals, finite sets of such sets and individuals, and so on.

DEFINITION. *Two models M' and M'' are discernible by means of finite sets if, and only if, there is a closed formula a in T^* such that*

$$\mathrm{stsf}_{M'}\, a \quad \textit{and} \quad \mathrm{stsf}_{M''} \sim a.$$

We define the sequence of games $H_n(M', M'')$. They are similar to the games $G_n(M', M'')$ except that, in every move player I points out an arbitrary finite number m of elements either from $|M'|$ or from $|M''|$.

As in the game G_n, player II has to match every individual pointed out by I with an individual in another model; the number of different elements chosen by both players need not be the same. After n moves we have $m_1 + m_2 + \ldots + m_n$ pairs of individuals.

$$\begin{array}{ll}
a'_{11} \longleftrightarrow a''_{11}, a'_{12} \longleftrightarrow a''_{12}, \ldots, a'_{1m_1} \longleftrightarrow a''_{1m_1} & \text{(1st move)} \\
a'_{21} \longleftrightarrow a''_{21}, a'_{22} \longleftrightarrow a''_{22}, \ldots, a'_{2m_2} \longleftrightarrow a''_{2m_2} & \text{(2nd move)} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
a'_{n1} \longleftrightarrow a''_{n1}, a'_{n2} \longleftrightarrow a''_{n2}, \ldots, a'_{nm_n} \longleftrightarrow a''_{nm_n} & (n\text{th move}).
\end{array}$$

The definition of winning is the same as for G_n.

THEOREM 2. *If player II has a winning method in $H_n(M', M'')$ for every n, then M' and M'' are indiscernible by finite sets.*

3. As an application of these theorems one can obtain the following results.

DEFINITION. *An element $x \in M$ is definable in the theory T**) for which M is a model, when there is a formula $a(\xi)$ in T such that*

$$\mathrm{stsf}_M\, a\binom{\xi}{y}$$

if, and only if, $y = x$.

*) It was pointed out to me by Professor A. Tarski that this theorem is but another formulation of Theorem 3 from [4].

**) Theory T can be non-elementary.

Let T_1 be the elementary theory of inequality $<$ for ordinal numbers, T_2 the elementary theory of addition $+$ for them, and T_3 the elementary theory of addition $+$ and multiplication $\cdot$. Let T_1^*, T_2^*, T_3^* be the corresponding non-elementary theories defined as in **2**. We denote the ordinals ω^ω, ω^{ω^ω}, and $\omega^{\omega^{\omega^\omega}}$ by x_1, x_2, x_3.

THEOREM 3. *The following three conditions are equivalent for every ordinal number x, and $i=1,2,3$:*

(a) *x is definable in T_i;*

(b) *x is definable in T_i^*;*

(c) $x < x_i$.

The equivalence of conditions (a)—(c) for $i=1$ follows immediately from results contained in [3]. The implication (c) $\rightarrow$ (a) was already known to Tarski who also raised the problem solved in Theorem 3.

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MATHÉMATIQUE

Sur un problème géométrique de P. Erdös

par

S. STRASZEWICZ

Présenté par K. BORSUK le 15 Octobre 1956

Cette note contient la solution — pour le cas de l'espace à 3 dimensions — du problème suivant, posé par P. Erdös [1], [2]. Envisageons dans l'espace euclidien à k dimensions les ensembles de n points $(n > k)$. Quel est le maximum $N_k(n)$ du nombre des couples de points d'un ensemble de cette classe tels que la distance entre les points de chaque couple soit égale au diamètre de cet ensemble?

Evidemment $N_1(n)=1$; on trouve facilement $N_2(n)=n$. Nous allons montrer que $N_3(n)=2n-2$.

Soit $A=\{a_1, a_2, \ldots, a_n\}$ un ensemble de $n>3$ points et de diamètre 1, situé dans l'espace à 3 dimensions et $N(A)$ le nombre des couples (a_i, a_k) de points de A tels que la distance $\varrho(a_i, a_k)=1$. Il est à démontrer que

$$N(A) \leqslant 2n-2.$$

Cette inégalité étant évidente pour $n=4$, supposons que $n>4$ et procédons par induction. Admettons que notre assertion soit vraie pour les ensembles de $n-1$ points. Désignons par μ_i $(i=1,2,\ldots,n)$ le nombre des points $a_k \in A$ tels que $\varrho(a_i, a_k)=1$. Alors

$$N(A)=\tfrac{1}{2}\sum_i \mu_i.$$

Deux cas peuvent se présenter.

1° Il existe un indice r tel que $\mu_r \leqslant 2$. Dans ce cas $N(A) \leqslant N(A-\{a_r\})+ +2 \leqslant 2(n-1)-2+2=2n-2$.

2° On a $\mu_i \geqslant 3$ pour $i=1,2,\ldots,n$. Soit K_i la sphère (fermée) de rayon 1 ayant a_i pour centre, S_i la surface de K_i. Posons $\prod_i K_i = Q$. L'ensemble Q est un corps convexe dont la surface $T=Q\sum_i S_i=\sum_i T_i$, où $T_i=QS_i$. Comme $A \subset Q$ et $A \subset \sum_i S_i$ on a $A \subset T$. On constate les faits suivants:

a) $AT_i = AQS_i = AS_i$, donc AT_i $(i=1,2,\ldots,n)$ est un ensemble de μ_i points;

b) chaque T_i est un domaine convexe sur la surface sphérique S_i.

On a, en effet,

$$T_i = QS_i = K_i S_i \prod_{j\neq i} K_j = S_i \prod_{j\neq i} K_j.$$

Or, $S_i K_j$ $(j\neq i)$ est une calotte sphérique limitée par une circonférence de rayon <1. L'ensemble T_i étant la partie commune des calottes $S_i K_j$ $(j=1,2,\ldots,i-1,i+1,\ldots,n)$ et contenant un ensemble de $\mu_i \geqslant 3$ points de A est par suite un ensemble convexe sur S_i qui ne se réduit ni à un point ni à un arc d'un grand cercle de S_i. T_i est donc un domaine convexe sur la surface S_i.

Envisageons le contour $R_i = T_i \sum_{k\neq i} S_k$ du domaine convexe T_i $(i=1,2,\ldots,n)$. Ceux des points de R_i qui appartiennent aux 3 surfaces S_k $(k=1,2,\ldots,n)$ au moins, forment un ensemble fini contenant l'ensemble AT_i. La courbe R_i est donc un polygone sphérique à $\mu_i+\nu_i$ sommets, où $\nu_i \geqslant 0$. La surface $T=\sum_i T_i$ est un polyèdre; les domaines T_i sont les faces de ce polyèdre, le nombre de ses arêtes est $\frac{1}{2}(\sum_i \mu_i + \sum_i \nu_i)$ et le nombre de ses sommets est $\leqslant n + \frac{1}{3}\sum_i \nu_i$. En vertu du théorème d'Euler, $\frac{1}{2}(\sum_i \mu_i + \sum_i \nu_i) \leqslant n + n + \frac{1}{3}\sum_i \nu_i - 2$, ce qui donne l'inégalité demandée.

$$N(A) = \tfrac{1}{2}\sum_i \mu_i \leqslant 2n-2-\tfrac{1}{6}\sum_i \nu_i \leqslant 2n-2.$$

Remarquons qu'il existe des ensembles A de n points pour lesquels $N(A)=2n-2$. Tel est par exemple l'ensemble des points $(\frac{1}{2},0,0)$, $(-\frac{1}{2},0,0)$, $(0,\sqrt{3}/2\cos ka/n-3, \sqrt{3}/2\sin ka/n-3)$, où $k=0,1,\ldots,(n-3)$ et $a=\arccos\frac{1}{3}$. Ainsi $N_3(n)=2n-2$, c. q. f. d.

POLYTECHNIQUE DE VARSOVIE

OUVRAGES CITÉS

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ASTRONOMY

Interpretation of the Longitude Effect in the Count of Galaxies

by

W. ZONN

Presented by W. DZIEWULSKI on September 27, 1956

In his well-known paper concerning the space distribution of galaxies E. Hubble [1] pointed out remarkable systematic variations in longitude of the counted number of galaxies in lower galactic latitudes. These variations cannot be ascribed to the space distribution of the galaxies and obviously must be interpreted as an influence of galactic absorption on the number of observed galaxies up to a uniform limiting magnitude. It seems reasonable to draw from this effect some quantitative conclusions concerning the distribution of absorbing matter in the vicinity of the Sun, and this is done in the present paper.

The distribution of the absorbing matter may be represented by means of equidensity surfaces which may be approximated, in the nearest vicinity of the Sun, by parallel planes. Let α be their inclination to the galactic equator, $l_0 \pm 90°$ galactic longitudes of intersection with the galactic equator of one of these planes passing through the Sun.

The mean density D of the absorbing matter may then be regarded as a function of Z:

$$Z = z - ax,$$

where x and z are the heliocentric galactic co-ordinates, x-axes being directed toward the point $l = l_0$, z-axes toward the galactic pole; $a = \operatorname{tg} \alpha$.

The mean total absorption A in a given direction amounts to

$$A = \int_0^\infty D(Z)\, dr = \frac{\cos b}{1 - a \operatorname{ctg} b \cos(l - l_0)} \int_0^\infty D(Z)\, dZ.$$

Let us denote $A_0 = \int_0^\infty D(Z)\, dZ$.

The mean absorption in latitude b amounts to

$$\bar{A}_b = \frac{A_0 \csc b}{\sqrt{1 - a^2 \operatorname{ctg}^2 b}} \simeq A_0 \csc b,$$

and may be approximated in this way if $a \operatorname{ctg} b \ll 1$, which condition is fulfilled in the present case because a is small and $b \geqslant 20°$. Thus,

$$A = \frac{A_0 \csc b}{1 - a \operatorname{ctg} b \cos (l - l_0)}. \tag{1}$$

In order to introduce into (1) the number N of galaxies per square degree to a uniform limiting magnitude m, Hubble's considerations have been slightly modified in the following way.

Let $m_1 = m - A$ be magnitudes free from galactic absorption and m_2 such magnitudes free from both galactic absorption and reddening effect:

$$m_1 - m_2 = f(m_1) = f(m - A). \tag{2}$$

Under assumption of uniform space distribution of galaxies we may write

$$\log N = 0.6[(m - A) - f(m - A)] + \text{const},$$

and for mean values,

$$\overline{\log N} = 0.6(m - \bar{A}) - \overline{f(m - A)} + \text{const}.$$

Developing $f(m - A)$ into series and neglecting second and higher exponents of $(A - \bar{A})$, we obtain

$$\overline{\log N} - \log N = 0.6(A - \bar{A})\left[1 - \frac{\mathrm{df}\,(m - \bar{A})}{dm}\right].$$

In order to evaluate the value of $\frac{\mathrm{df}\,(m - \bar{A})}{dm}$ we assume, in accordance with Hubble, $f(m_1)$ to be proportional to the distances of galaxies thus getting the relation

$$\log f(m_1) = 0.2[m_1 - f(m_1)] + \text{const},$$

from which we may easily calculate the value of $\frac{\mathrm{df}\,(m - \bar{A})}{dm}$.

Although it obviously depends upon galactic latitude (as $\bar{A}$ is a function of b), yet it varies so slowly with b that we may assume

$$\frac{\mathrm{df}\,(m - \bar{A})}{dm} = \text{const} = 0.117,$$

the numeric value being evaluated from the data given by Hubble. Thus we finally obtain for a given galactic latitude

$$(\overline{\log N})_b - (\log N)_b = 0.53 \frac{A_0 a \csc b \operatorname{ctg} b \cos (l - l_0)}{1 - a \operatorname{ctg} b \cos (l - l_0)}. \tag{3}$$

Using the data contained in Hubble's paper we obtain a set of equations (3) with the unknowns a and l_0 which may be easily found by the least squares method for different galactic latitudes. The results are given in the following table, together with their mean errors:

By means of the relation (3) we may easily explain why the longitude effect appears only in low galactic latitudes, $|b| < 45°$. The amplitude of variations of $\overline{\log N} - \log N$ is, at first approximation, proportional to $\operatorname{ctg} b \csc b$ (since a is small). This quantity decreases very rapidly with increasing b and in high galactic latitudes becomes practically zero. Therefore we limited our calculations only to lower galactic latitudes up to $|b| = 20°$, which corresponds to the beginning of the zone of avoidance.

TABLE

b	a	l_0
$+45°$	0.12 ± 0.14	$246° \pm 37°$
40	0.20 0.12	359 34
35	0.29 0.11	38 31
30	0.17 0.07	354 23
25	0.25 0.04	328 16
20	0.12 0.08	337 18
$-20°$	0.14 0.05	2 22
25	0.21 0.06	326 24
-30	0.27 0.09	345 37
Weighted means	0.20 ± 0.02	$338° \pm 10°$

The present results show that the mean density of interstellar matter (or more exactly its solid component which is responsible for interstellar absorption) runs in accordance with the mean density of stars in high galactic latitudes as it has been shown by J. Oort [2] and later by S. Habibullin [3]. The equidensity surfaces of stars in the vicinity of the Sun are clearly inclined towards the galactic equator as it may be easily seen on Figs. 1 and 2 (reproduced from the papers of the writers mentioned), where the equidensity planes of interstellar matter are shown by thick dotted lines.

According to Habibullin the inclination amounts to 13°; in the present case we obtain $\alpha = 12°$ which is in good agreement with the value mentioned above.

If we assume that the density of interstellar matter in the direction perpendicular to the galactic plane may be represented by the formula

$$D(Z) = D(0) \exp\left(-\frac{|Z|}{\beta}\right) \qquad \beta = 100 \text{ ps},$$

as it was first suggested by E. Williams [4] and grounded by P. Parenago [5] and P. J. van Rhijn [6], we may easily obtain the logarithmic density gradient in the galactic plane (along the line from the centre of the Galaxy to the Sun)

$$\frac{\partial \log D}{\partial R} = -0.91 \text{ kps}^{-1},$$

which is about ten times greater than that of the stars belonging to the plane component of the Galaxy and about four times greater than the

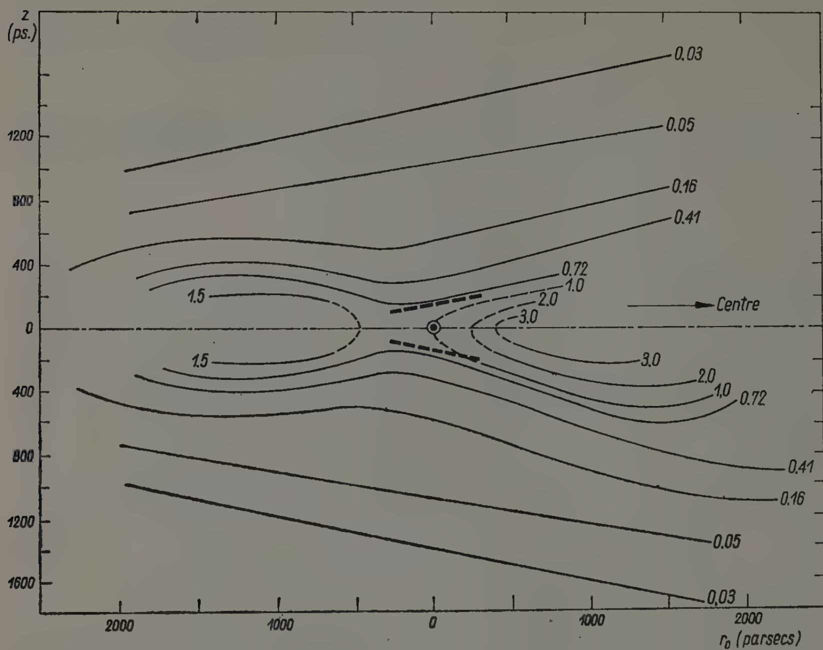
z
(ps.)
1200
800
400
0
400
800
1200
1600
0.03
0.05
0.16
0.41
0.72
1.0
2.0
3.0
1.5
Centre
3.0
2.0
1.0
0.72
1.5
0.41
0.16
0.05
0.03
2000
1000
0
1000
2000
r_0 (parsecs)

Fig. 1

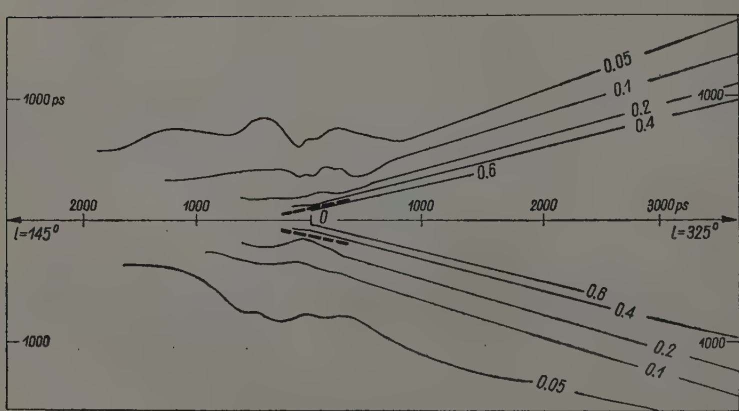
1000 ps
0.05
0.1
0.2
0.4
1000
0.6
2000
1000
0
1000
2000
3000 ps
l=145°
l=325°
0.6
0.4
0.2
0.1
0.05
1000
1000

Fig. 2

logarithmic density gradient of stars belonging to the spherical component. Yet it is worth while to stress that this great absolute value corresponds only to the relatively small region surrounding the Sun.

The present results suggest that the solid component of interstellar matter forms a spiral arm in the neighbourhood of the Sun as it has been established in the case of the gaseous component of interstellar matter and of the early type stars.

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ASTRONOMY

On Some Consequences of the Patchy Structure of Interstellar Matter

by

W. ZONN

Presented by W. DZIEWULSKI on September 27, 1956

We consider the selection factor due to the patchy structure of interstellar matter, which systematically lessens the observed colour-excesses of stars, especially of those of apparent magnitudes near the limiting magnitude of catalogue. It may be taken into account treating the number x of clouds intersected by the line of sight as a random variable ruled by the probability $p(x)$. Then the observed mean colour-excess $\bar{E}$ of a group of stars of the same distance may be expressed by

$$\bar{E} = \varepsilon \frac{\sum_{0}^{x_0} x\, p(x)}{\sum_{0}^{x_0} p(x)}, \tag{1}$$

where ε is the mean colour-excess of a single cloud, x_0 — the maximum number of clouds.

If $p(x)$ is Poisson's distribution (which may be supposed since the number of clouds is rather small) the relation (1) may be rewritten:

$$\bar{E} = \varepsilon\lambda \frac{\sum_{0}^{x_0-1} p(x)}{\sum_{0}^{x_0} p(x)} = E_0 \left[1 - \frac{p(x_0)}{\sum_{0}^{x_0} p(x)}\right], \tag{2}$$

where λ is the mean number of clouds intersected by the line of sight in a given distance. $E_0 = \varepsilon\lambda$.

The value of x_0 may be obtained by means of "corrected" magnitudes m' of single stars (free from interstellar absorption):

$$m' = m - \gamma E = m - \gamma\varepsilon x,$$

where γ is the ratio of general to selective absorption, m — the observed magnitude. Grouping stars according to their $m's$ we have:

$$m \leqslant m_0$$

(m_0 being the limiting magnitude of a given catalogue), and since

$$x_0 = \frac{m_0 - \overline{m}'}{\varepsilon\gamma}. \tag{3}$$

Relations (2) and (3) enable to determine the corrected mean colour-excess for given $\bar{E}$ assuming γ and $\varepsilon\gamma$ to be known. Equation (2) may be solved by the method of successive approximations (adopting as the first one $E_0 = \bar{E}$), or by means of numeric tables which may be easily constructed from data given in statistical tables. The corresponding distance r may be found from the relation:

$$\log r = 0.2\ (m' - M) + 1,$$

where M is the absolute magnitude of the selected group of stars.

Applying this method to the data contained in T. Elvius' Catalogue [1] under assumption of $\gamma = 4$ and $\varepsilon = 0\overset{m}{.}25/\gamma = 0\overset{m}{.}0625$, we obtain the relations of $E_0(r)$ and from them the mean colour-excess $E_0(z)$ as a function of the distance z from the galactic plane, under assumption that the mean density of the interstellar matter depends upon z only. In other words,

$$E_0(r) \sin b = E_0(z),$$

where b is the galactic latitude of a given direction. From fifteen Selected Areas we obtain the mean $E_0(z)$ curve which fits well in the relation:

$$E_0(z) = B(1 - e^{-\frac{z}{\beta}}),$$

first found by E. Williams [2] and later grounded and improved by P. Parenago [3] and P. J. van Rhijn [4]. The constants in this formula found by the least squares method amount to:

$$B = 0\overset{m}{.}0645 \pm 0\overset{m}{.}0007,$$

$$\beta = 125 \pm 2\,ps,$$

the value β being somewhat greater than usually obtained because of modifying the method of its determination.

Calculating further the extrapolated $E_0(r)$ values for $r = \infty$ in each direction and comparing them with the logarithms of numbers of galaxies contained in Hubble's well known paper [5] we obtain the following empirical relation (assumed to be linear):

$$E_0(\infty) = 0.42\ (2.07 - \log N).$$

Comparing this with J. Oort's formula [6] for photographic absorption A:

$$A=\frac{2.17-\log N}{0.55},$$

we finally obtain

$$\gamma=4.3\pm1.2.$$

Analysis of deviations of a single value from the mean values of corrected colour-excesses in different directions gave some information on the mean colour-excess in a cloud of interstellar matter and the dispersion of this value.

Starting from the fact that the observed value is the colour index $CI=CI_0+\sum_0^n e_j$ (where CI_0 is the intrinsic colour index of a given spectral subclass, e_j — the colour excess in a single cloud), and assuming that the number n of clouds intersected by the line of sight is independent from e_j we may obtain:

$$\sigma^2=\sigma_0^2+E_0\left(\varepsilon+\frac{\sigma_e^2}{\varepsilon}\right),$$

where σ is the observed dispersion, σ^2 — the dispersion caused by observational errors and the dispersion of colour-indexes of stars belonging to the same spectral subclass, σ_e — the dispersion of colour excesses in clouds of interstellar matter, ε — the mean colour excess in a cloud, and E_0 — the mean colour excess of group of stars at the same distance. From the values of E_0 and σ^2 obtained we find:

$$\varepsilon+\frac{\sigma_e^2}{\varepsilon}=0^m.105\pm0^m.014,$$

$$\sigma_0^2=0^m.010\pm0.003.$$

If we assume $\varepsilon=0^m.25/\gamma=0^m.058$ we obtain

$$\sigma_e^2=0^m.003,$$

which is in contradiction to the hypothis of P. Parenago [3] as to the Maxwellian distribution of absorption in interstellar clouds. This distribution leads to the relation:

$$\frac{\sigma_e^2}{\varepsilon}=0.18\,\varepsilon,$$

when in the present case we have

$$\frac{\sigma^2}{\varepsilon}\simeq\varepsilon.$$

Better agreement can be obtained if we assume, in accordance with D. Chawtasi's suggestion [7] that the distribution of masses of clouds

as well as of their dimensions are decreasing functions of corresponding arguments. If we adopt the distribution of colour-excess e in clouds as follows:

$$f(e) = C e^{-n},$$

we obtain $\frac{\sigma_e^2}{\varepsilon} \simeq \varepsilon$ under assumption that $n = 0.6$ (adopting some minimal value of e different from zero).

The details of calculations as well as the representation of full data will be published elsewhere [8].

It is worth notiing that the disregard of the selection factor introduced by the patchy structure of the interstellar matter could be sources of systematic errors in other investigations. As an example we may consider Schwarzschild's well known integral equation of stellar statistics which in the present case is to be written

$$A(m) = \omega \int_0^\infty D(r) r^2 dr \int_0^\infty \varphi(m - 5 \log r + 5 - a_0 x) p(x) dx,$$

where the symbols are those usually used: a_0 is the mean absorption in a cloud, x — the number of intersected clouds ruled by the probability $p(x)$.

The second integral is the mean value of the luminosity function which differs from

$$\varphi(m - 5 \log r + 5 - a_0 \bar{x}),$$

used previously in its place.

The writer wishes to express his sincere thanks to Professor J. Łukaszewicz and to A. Kruszewski for their valuable suggestions and hints.

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THEORETICAL PHYSICS

On a Further Modification of Dirac's δ-Functions

by

L. INFELD and J. PLEBAŃSKI

Presented by L. INFELD on October 19, 1956

1. Introduction. In a previous paper [1] we developed the theory of modified (three-dimensional) Dirac functions denoted by $\hat{\delta}(\vec{x})$. Axiomatically those functions were defined as follows:

($\hat{A}_1$): $\hat{\delta}(\vec{x})$ can be treated formally as a spherically symmetric function for which all the derivatives exist.

($\hat{A}_2$): $\hat{\delta}(\vec{x})=0$ for $\vec{x}\neq 0$.

($\hat{A}_3$): for every continuous $f(\vec{x})$ in the neighbourhood of $\vec{x}_0, \Omega(\vec{x}_0)$, we have

$$\int\limits_{\Omega(\vec{x}_0)} d_3x\hat{\delta}(\vec{x}-\vec{x}_0)f(\vec{x})=f(\vec{x}_0).$$

($\hat{A}_4$): for an arbitrary neighbourhood $\Omega(0)$ of the point $\vec{x}=0$, we have

$$\int\limits_{\Omega(0)} d_3x\hat{\delta}(\vec{x})|\vec{x}|^{-p}=0 \quad (p=1,2,...,k).$$

The self-consistency of these axioms was shown by the existence of a class of sequences $\{\hat{\delta}_n(\vec{x})\}$ for which $\hat{A}_1$-$\hat{A}_4$ were valid in the limit $n\to\infty$. In [1] we described the manner of constructing such sequences with the help of the Jacobian polynomials.

2. A simple model of $\hat{\delta}_n(\vec{x})$. We shall show here, first, that it is possible to simplify essentially our previous model of $\{\hat{\delta}_n(\vec{x})\}$. First, instead of dealing with a sequence $\hat{\delta}_n(\vec{x})$, it will be more convenient to deal with a continuous parameter $\hat{\delta}_\varepsilon(\vec{x})$, so that for $\varepsilon\to 0$ the axioms $\hat{A}_1$-$\hat{A}_4$ will be valid. Let us start with a model of an ordinary Dirac δ-function but satisfying the following conditions:

$$\delta_\varepsilon(\vec{x})=\varepsilon^{-3}\Delta\left(\frac{|\vec{x}|}{\varepsilon}\right), \tag{2.1}$$

where $\Delta(z)$ is such that

$$(2.2)\qquad D_p \underset{\mathrm{df}}{=} 4\pi \int_0^\infty dz z^{2-p}\Delta(z) \equiv \int_\infty d_3z \Delta(|\vec{z}|)|\vec{z}|^{-p}, \qquad (p=1,2,\dots,k)$$

always exists. We also assume

$$D_0=1, \qquad D_p \neq 0, \qquad (p=1,2,\dots,k).$$

Furthermore, we assume that

$$(2.3)\qquad \Delta(z) \in C_\infty \quad \text{for} \quad z \in [0, \infty),$$

and that for large z, $\Delta(z)$ and its derivatives reduce to zero at least exponentially. This means that for every n there exists a $z_0>0$ and an $a>0$, $b>0$, such that, for every $z>z_0$, we have:

$$(2.4)\qquad ae^{-bz} > |\Delta^{(n)}(z)|.$$

For practically every model of an ordinary Dirac δ_ε-function we can find a model of a δ_ε-function satisfying (2.1)-(2.4). Indeed, the only condition usually not satisfied is (2.2). But if the chosen $\Delta(z)$ function does not have this property, it can be made to have it by multiplying it by z^{k-2} and renormalising. For example, $\delta_\varepsilon(\vec{x})=(2\pi)^{-3/2}\varepsilon^{-3}e^{-1/2|\vec{x}|^2\varepsilon^{-2}}$ has all the above properties with the exception of (2.2), in this case $\Delta(z)=(2\pi)^{-3/2}e^{-1/2z^2}$. But the function

$$\delta_\varepsilon(\vec{x})=\frac{1}{2\pi 2^{\frac{k+1}{2}}\Gamma\left(\frac{k+1}{2}\right)}\frac{1}{\varepsilon^3}\left(\frac{|\vec{x}|}{\varepsilon}\right)^{k-2}e^{-1/2|\vec{x}|^2\varepsilon^{-2}}$$

also has the property (2.2). The D_p's are:

$$D_p=2^{-p/2}\frac{\Gamma\left(\frac{k-p+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)}.$$

Now, having the $\Delta(z)$ satisfying (2.2)-(2.4), and therefore having the model of an "ordinary" Dirac function, $\hat{\delta}_\varepsilon(\vec{x})$, we can give a simple example of the $\hat{\delta}_\varepsilon(\vec{x})$ function. This is:

$$(2.5)\qquad \boxed{\hat{\delta}_\varepsilon(\vec{x})=\hat{T}_\varepsilon\delta_\varepsilon(\vec{x}),}$$

where $\hat{T}_\varepsilon$ is a simple operator:

$$(2.6)\qquad \boxed{\hat{T}_\varepsilon=\frac{1}{k!}\left(\frac{\partial}{\partial\varepsilon}\right)^k\varepsilon^k.}$$

Indeed, $\hat{A}_1$ is satisfied for every ε in $\hat{\delta}_\varepsilon(\vec{x})$. Then, $\hat{A}_2$ is satisfied in the limit for $\varepsilon \to 0$. (Assumption 2.4). Let us prove $\hat{A}_4$.

$$\int\limits_{\infty} \hat{\delta}_\varepsilon(\vec{x})\,|\vec{x}|^{-p} d_3x = \frac{1}{k!}\left(\frac{\partial}{\partial\varepsilon}\right)^k \varepsilon^{k-p} \int\limits_0^\infty \Delta(z) z^{2-p} dz = \tag{2.7}$$

$$= \frac{1}{k!}\left(\frac{\partial}{\partial\varepsilon}\right)^k \varepsilon^{k-p} D_p = \begin{cases} 1 \text{ for } p=0 & \text{(a)} \\ 0 \text{ for } p=1,2,\ldots,k. & \text{(b)} \end{cases}$$

From (2.7) and $\hat{A}_2$ we see that $\hat{A}_4$ is satisfied in the limit $\varepsilon \to 0$. $\hat{A}_3$ follows from (2.7a) in the same way as it follows for models of ordinary Dirac functions, of which the $\hat{\delta}_\varepsilon$ are only special cases. Indeed, (2.7b) does not play a role in the proof of $\hat{A}_3$ as long as $f(\vec{x})$ is *continuous* in the neighbourhood of $\vec{x}_0$.

3. On $\hat{\hat{\delta}}$ functions. Let us now define $\hat{\hat{\delta}}(\vec{x})$ similar to $\delta(\vec{x})$ but with one axiom changed. Into the axioms $\hat{A}_1$, $\hat{A}_2$, $\hat{A}_3$ we introduce $\hat{\hat{\delta}}$ instead of $\hat{\delta}$ (calling them $\hat{\hat{A}}_1$, $\hat{\hat{A}}_2$, $\hat{\hat{A}}_3$) and

$(\hat{\hat{A}}_4)$: $$\int\limits_{\Omega(0)} d_3x\, \hat{\hat{\delta}}(\vec{x})|\vec{x}|^{-p} = \omega_p, \qquad (p=1,2,\ldots,k) \qquad (\omega_0 \underset{\text{df}}{=} 1),$$

where ω_p are *prescribed* numbers. Again, we shall show the consistency of $\hat{\hat{A}}_1 - \hat{\hat{A}}_4$ by forming a proper model of the $\hat{\hat{\delta}}_\varepsilon(\vec{x})$ -function satisfying the axioms $\hat{\hat{A}}_1 - \hat{\hat{A}}_4$ within the limit $\varepsilon \to 0$.

Similarly as in **2**, we first introduce a $\delta_\varepsilon(\vec{x})$ which satisfies (2.1)–(2.4); then we define

$$\boxed{\hat{\hat{\delta}}_\varepsilon(\vec{x}) = \hat{\hat{T}}_\varepsilon \delta_\varepsilon(\vec{x})}, \tag{3.1}$$

$$\boxed{\hat{\hat{T}}_\varepsilon = \frac{1}{k!}\sum_{s=0}^{k} (-1)^s \frac{\omega_s}{D_s}\binom{k}{s}\left(\frac{\partial}{\partial\varepsilon}\right)^{k-s} \varepsilon^{k-s}\left(\varepsilon^2\frac{\partial}{\partial\varepsilon}\right)^s}. \tag{3.2}$$

First we see that $\hat{\hat{T}}_\varepsilon$ is a generalisation of $\hat{T}_\varepsilon$ and goes over into $\hat{T}_\varepsilon$ for $\omega_p = 0$, $p=1,2,\ldots,k$.

We shall now show that $\hat{\hat{A}}_4$ is satisfied for every ε; therefore it must also be satisfied within the limit $\varepsilon \to 0$:

$$\int\limits_{\infty} d_3x\, \hat{\hat{\delta}}_\varepsilon(\vec{x})|\vec{x}|^{-p} = \hat{\hat{T}}_\varepsilon \varepsilon^{-p} \int\limits_0^\infty dz\, z^{2-p}\Delta(z) = D_p \hat{\hat{T}}_\varepsilon \varepsilon^{-p}.$$

We know that

$$(-1)^s\left(\varepsilon^2\frac{\partial}{\partial\varepsilon}\right)^s \varepsilon^{-p} = \left(\frac{\partial}{\partial\varepsilon^{-1}}\right)^s (\varepsilon^{-1})^p = \begin{cases} 0 \text{ for } s > p \\ s!\dbinom{p}{s}\varepsilon^{s-p} \text{ for } s \leq p. \end{cases}$$

Therefore

$$\hat{\hat{T}}_s \varepsilon^{-p} = \frac{1}{k!} \sum_{s=0}^{p} \frac{\omega_s}{D_s} s! \binom{k}{s} \binom{p}{s} \left(\frac{\partial}{\partial \varepsilon}\right)^{k-s} \varepsilon^{k-p}.$$

We know that

$$\left(\frac{\partial}{\partial \varepsilon}\right)^{k-s} \varepsilon^{k-p} = \begin{cases} 0 \text{ for } s<p \\ (k-p)! \text{ for } s=p. \end{cases}$$

Therefore, finally,

$$\int\limits_{\infty} d_3x\, \hat{\hat{\delta}}_s(\vec{x})\, |\vec{x}|^{-p} = D_p \hat{\hat{T}}_\varepsilon \varepsilon^{-p} = D_p \frac{\omega_p}{D_p} \binom{k}{p} \binom{p}{p} \frac{p!(k-p)!}{k!} = \tag{3.2}$$

$$= \begin{cases} 1 \quad \text{for } p=0 & \text{(a)} \\ \omega_p \text{ for } p=1,2,\ldots,k & \text{(b)} \end{cases}$$

The proof that $\hat{\hat{\mathrm{A}}}_1, \hat{\hat{\mathrm{A}}}_2, \hat{\hat{\mathrm{A}}}_3$ are satisfied in the limit follows exactly as in the case of ordinary Dirac functions, because (3.2b) does not play any role in this proof and the $\hat{\hat{\delta}}_\varepsilon$ are ordinary models of Dirac functions.

In other words, Dirac functions are characterised by the values of $\omega_1, \ldots, \omega_k$. Usually it is understood that all of them are infinite ($\omega_p = \infty$, $p = 1, 2, \ldots$). However, this need not be the case. An arbitrary but finite number of these can be made finite. If $\omega_1, \ldots, \omega_k$ equal zero, we can avoid the process of renormalisation.

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THEORETICAL PHYSICS

The Representation Leading to Isobars of the Nucleon in the Fixed-Source Theory

by

W. KRÓLIKOWSKI

Presented by W. RUBINOWICZ on October 29, 1956

This paper returns to the problem of isobars of the nucleon in the fixed source theory [1].

In this paper it is shown that in the symmetrical pion theory with slowly moving source representing the nucleon there exist four isobars of the physical nucleon, if an approximation procedure is applicable. This procedure differs from the strong coupling approximation [1] and depends on splitting the interaction Hamiltonian in two parts, the first of which gives four "potentials" that lead together with the free Hamiltonian to four sets of states, containing as ground states the ground states of four stable isobars of the physical nucleon. The second part — if it is small — causes the transitions between these four sets of states.

We consider the interacting system consisting of one nucleon and of pion field. The state equation of the system has the form

$$\left[i\hbar\frac{\partial}{\partial t} - H^N(\vec{p}) - H^M(\varphi,\pi) - H^{\text{int}}(\vec{x},\varphi)\right]\Psi(t) = 0, \tag{1.1}$$

where H^N, H^M, H^{int} are the energies of nucleon, pions and of interaction respectively. We have

$$H^M(\varphi,\pi) = \tfrac{1}{2}\int d_3x[\pi_j^3 + c^2(\partial_i\varphi_i)^2 + c^2\mu^2\varphi_j^2] = \tfrac{1}{2}\int d_3x[\pi_j^2 + c^2\varphi_j(-\Delta + \mu^2)\varphi_j], \tag{1.2}$$

where

$$[\pi_j(\vec{x}),\ \varphi_{j'}(\vec{x}\,')] = -i\hbar\delta_{jj'}\delta(\vec{x}-\vec{x}\,'),$$

and, if we take into account the symmetrical pion theory with slowly moving extended source

$H^N(\vec{p}) = \vec{p}^2/2M$ (after Mc^2 is transformed from the Hamiltonian),

$$
\begin{aligned}
H^{\text{int}}(\vec{x},\varphi) &= g\sigma_i\tau_j\int d_3x'\varrho(\vec{x}-\vec{x}')\partial_i'\varphi_j(\vec{x}') = -g\sigma_i\tau_j\int d_3x'\partial_i'\varrho\varphi_j = \\
&= \sigma_i\tau_j u_{ij}(\vec{x},\varphi), \\
u_{ij}(\vec{x},\varphi) &= g\int d_3x'\varrho\partial_i'\varphi_j.
\end{aligned} \tag{1.3}
$$

In the fixed-source limit we have

$$
\begin{aligned}
H^N &= 0, \\
H^{\text{int}}(\varphi) &= g\sigma_i\tau_j\int d_3x\varrho(\vec{x})\partial_i\varphi_j(\vec{x}) = -g\sigma_i\tau_j\int d_3x\partial\varrho\varphi_j = \\
&= \sigma_i\tau_j u_{ij}(\varphi), \\
u_{ij}(\varphi) &= g\int d_3x\varrho\partial_i\varphi_j.
\end{aligned} \tag{1.3'}
$$

As is well known [1], the interaction energy (1.3) may be written in the form

$$
H^{\text{int}}(\vec{x},\varphi) = \sigma_i\tau_j u_{ij}(\vec{x},\varphi) = \sigma_i'\tau_i'\underline{u}_i(\vec{x},\varphi) = U\sigma_i\tau_i U^{-1}\underline{u}_i(\vec{x},\varphi), \tag{1.4}
$$

where

$$
\sigma_i' = A_{ij}\sigma_j = U\sigma_i U^{-1}; \qquad \tau_j' = \tau_i B_{ij} = U\tau_j U^{-1}, \tag{1.5}
$$

$A=(A_{ij})$, $B=(B_{ij})$ being orthogonal matrices, U — unitary operator. A, B and thus also $\vec{\sigma}'=A\cdot\vec{\sigma}$, $\vec{\tau}'=\vec{\tau}\cdot B$ depend on $\vec{x}$, φ; u depends on $\vec{x}$, φ and in general on $\vec{\sigma}$, $\vec{\tau}$. In (1.4) $\underline{u}_i(\vec{x},\varphi)$ $(i=1,2,3)$ are roots of the secular equation:

$$
-\underline{u}_i^6 + S(\vec{x},\varphi)\underline{u}_i^4 - M(\vec{x},\varphi)\underline{u}_i^2 + D(\vec{x},\varphi) = 0, \tag{1.6}
$$

where

$$
\left\{
\begin{aligned}
S(\vec{x},\varphi) &= \operatorname{sp}(uu^T) = \sum_{ij=1}^{3} u_{ij}^2, \\
D(\vec{x},\varphi) &= \det(uu^T), \\
M(\vec{x},\varphi) &= \sum_{i=1}^{3}\,[\text{minor of } D \text{ corresponding to } (uu^T)_{ii}]
\end{aligned}
\right. \tag{1.7}
$$

and $u(\vec{x},\varphi)=(u_{ij}(\vec{x},\varphi))$, T denotes the transposed matrix. We take

$$
\underline{u}_i(\vec{x},\varphi) = \pm\sqrt{\underline{u}_i^2(\vec{x},\varphi)} \qquad \text{if} \qquad \det u \gtrless 0. \tag{1.8}
$$

We obtain [1] formulae (1.4)—(1.8) diagonalizing the symmetrical matrix uu^T. We have $\underline{u}^2=(\underline{u}_i^2\delta_{ij})=Auu^TA^T$ and $\underline{u}=(\underline{u}_i\delta_{ij})=AuB$, where $A^T=A^{-1}$, $B^T=B^{-1}$.

We have, of course,

$$(1.9)\qquad \begin{cases} S(\vec{x},\varphi)=\sum\limits_{i=1}^{3} \underline{u}_i^2, \\ D(\vec{x},\varphi)=\underline{u}_1^2\underline{u}_2^2\underline{u}_3^2, \\ M(\vec{x},\varphi)=\underline{u}_2^2\underline{u}_3^2+\underline{u}_3^2\underline{u}_1^2+\underline{u}_1^2\underline{u}_2^2. \end{cases}$$

Now we split the interaction energy (4) as follows

$$(1.10)\qquad H^{\text{int}}(\vec{x},\varphi)=H_1^{\text{int}}(\vec{x},\varphi)+H_2^{\text{int}}(\vec{x},\varphi),$$

where

$$(1.11)\qquad \begin{cases} H_1^{\text{int}}(\vec{x},\varphi)=\sigma_i\tau_i\underline{u}_i(\vec{x},\varphi) \\ H_2^{\text{int}}(\vec{x}.\varphi)=(U\sigma_i\tau_iU^{-1})\underline{u}_i(\vec{x},\varphi). \end{cases}$$

We note that operators $\sigma_i\tau_i$ $(i=1,2,3)$ commute.

The standard ket of the system may be written as the following product: $\rangle=\rangle^N\rangle^M=\rangle^p\rangle^s\rangle^M$, where $\rangle^N=\rangle^p\rangle^s$ and $\rangle^M$ are the standard kets pertaining to degrees of freedom of nucleon and pions respectively. $\rangle^p$ and $\rangle^S$ – pertaining respectively to position and spin and isotopic spin degrees of freedom of the nucleon.

We take into account the eigenvalue problem for the commuting operators $\sigma_i\tau_i$ $(i=1,2,3)$:

$$(1.12)\qquad \sigma_i\tau_i|a\rangle^s=e_i^{(a)}|a\rangle^s.$$

We get [1] four simultaneous eigenstates $|a\rangle^s$ $(a=1,2,3,4)$ and eigenvalues

$$(1.13)\qquad e_i^{(a)}=\pm 1 \qquad e_1^{(a)}e_2^{(a)}e_3^{(a)}=-1.$$

Since $(\vec{\sigma}+\vec{\tau})^2$ commutes with $\sigma_i\tau_i$ $(i=1,2,3)$, one of the states $|a\rangle^s$, say $|a\rangle^s$ $(a=1)$, is a singulet, other, $|a\rangle^s$ $(a=2,3,4)$, are triplets with respect to addition of vectors $\vec{\sigma}$ and $\vec{\tau}$ from two different states.

From (1.1), (1.10), (1.11), (1.12) and (1.13) we obtain

$$(1.14)\qquad \left[i\hbar\frac{\partial}{\partial t}-H^N(\vec{p})-H^M(\varphi,\pi)-V_a(\vec{x},\varphi)\right]\Psi(a,t)= \\ =\sum_{a'}{}^s\langle a|H_2^{\text{int}}(\vec{x},\varphi)|a'\rangle^s\Psi(a',t),$$

where $\Psi(a,t)=\Psi(a,t)\rangle^p\rangle^M={}^s\langle a|\Psi(t)\rangle$ is the wave function with respect to spin and isotopic spin degrees of freedom of the nucleon in the representation $|a\rangle^s$, and

$$V_a(\vec{x},\varphi)=e_i^{(a)}\underline{u}_i(\vec{x},\varphi)\qquad (a=1,2,3,4)$$

or writing explicitly

$$\left\{\begin{aligned} V_1(\vec{x},\varphi) &= -\underline{u}_1-\underline{u}_2-\underline{u}_3,\\ V_2(\vec{x},\varphi) &= -\underline{u}_1+\underline{u}_2+\underline{u}_3,\\ V_3(\vec{x},\varphi) &= \quad\underline{u}_1-\underline{u}_2+\underline{u}_3,\\ V_4(\vec{x},\varphi) &= \quad\underline{u}_1+\underline{u}_2-\underline{u}_3.\end{aligned}\right. \tag{1.15}$$

We can see that equation (1.14) describes – if matrix elements of $H_2^{\mathrm{int}}(\vec{x},\varphi)$ are small – the "motions" of the system interacting by one of four "potentials" $V_a(\vec{x},\varphi)$ and transitions of the system between states corresponding to different "potentials". In this case one may speak of four sets a of states communicating with each other by means of transitions caused by the interaction $H_2^{\mathrm{int}}(\vec{x},\varphi)$. There are, of course, also transitions between states belonging to the same set a, if we define these states not by means of constants of motion. Ground states in four sets a may be interpreted as ground states of four isobars of the physical nucleon. Some of the excited states in the four sets a may be considered as excited states of these four isobars, others as states of four isobars and of mesons.

We have here two types of transitions: 1) between different sets a, caused by the interaction $H_2^{\mathrm{int}}(\vec{x},\varphi)$, and 2) within the set a, caused by an interaction that we shall obtain in the next paper.

As yet just four types of barions (N, Λ, Σ, Ξ) stable in "fast" transitions are known. They communicate with each other by "slow" transitions. In addition to them the "isobar" in Brueckner's sense is known to occur as real intermediate state in scattering processes of pions on nucleons. It communicates with the nucleon by "fast" transitions. Thus, we have experimentally just two types of transitions.

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THEORETICAL PHYSICS

The Separation of Non-Interacting Pion Degrees of Freedom in the Fixed-Source Theory

by

W. KRÓLIKOWSKI

Presented by W. RUBINOWICZ on October 29, 1956

In this paper *) the system of all oscillators of the pion field is split into oscillators of three kinds: interacting "directly" with the fixed extended source representing the nucleon, interacting "indirectly", and non-interacting. There are nine oscillators interacting directly with the nucleon. In the representation used in the first section, they interact mutually by means of the second part of the interaction Hamiltonian. The system of these nine oscillators together with the source is proposed as an approximate model for the four isobars of the physical nucleon. Transitions between these states, inside the same isobar, are caused by the interaction operator obtained from a part of the free Hamiltonian. Thus, there are in this theory two types of transitions given by two different operators. If the second part of the interaction Hamiltonian is small, transitions between different isobars have distinctly smaller probabilities than transitions within the same isobar.

The results of this and the former paper depend on the smallness of matrix elements of the second part of the interaction Hamiltonian. So far no accurate solution for the proposed model of four isobars has been obtained, which would enable us to investigate the smallness of matrix elements of the second part of the interaction Hamiltonian. An approximation procedure is given in the subsequent paper.

We take into account the fixed-source symmetrical pseudoscalar theory:

Using the expansion in spherical harmonics $Y_l^m(\vartheta,\varphi)=e^{im\varphi}P_l^m(\cos\vartheta)$ [1], we have

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*) This paper is the sequel of the paper *The Representation Leading to Isobars of the Nucleon in the Fixed-Source Theory.*

$$(2.1)\quad \begin{cases} \varphi_j(\vec{x}) = \sum\limits_{lm} \varphi_{lm}^{(j)}(r) \left[\dfrac{2l+1}{4\pi}\dfrac{(l-m)!}{(l+m)!}\right]^{1/2} Y_l^m(\vartheta,\varphi), \\ \pi_j(\vec{x}) = \sum\limits_{lm} \pi_{lm}^{(j)}(r) \left[\dfrac{2l+1}{4\pi}\dfrac{(l-m)!}{(l+m)!}\right]^{1/2} Y_l^m(\vartheta,\varphi), \end{cases}$$

where

$$\int \sin\vartheta\, d\vartheta\, d\varphi\, Y_l^{m*}(\vartheta,\varphi)\, Y_{l'}^{m'}(\vartheta,\varphi) = \delta_{ll'}\,\delta_{mm'}\,\frac{4\pi}{2l+1}\,\frac{(l+m)!}{(l-m)!}$$

and

$$[\pi_{lm}^{(j)}(r),\, \varphi_{l'm'}^{(j')}(r')] = -i\hbar\,\delta_{jj'}\,\delta_{ll'}\,\delta_{mm'}\,\frac{\delta(r-r')}{rr'}.$$

The Hamiltonians (1.2) and (1.3′) take now the forms

$$(2.2)\quad H^M(\varphi,\pi) = \sum_l H_l^M(\psi,\pi) =$$

$$= \sum_{lm} \frac{1}{2}\int_0^\infty r^2 dr \left[\pi_{lm}^{(j)*}\pi_{lm}^{(j)} + \varphi_{lm}^{(j)*}\omega_l^2\left(r,\frac{d}{dr}\right)\varphi_{lm}^{(j)}\right] =$$

$$= \sum_{i\neq 1} H_l^M(\varphi,\pi) + \frac{1}{2}\int_0^\infty r^2 dr \left[\pi_{ij}^2 + \varphi_{ij}\omega_1^2\left(r,\frac{d}{dr}\right)\varphi_{ij}\right],$$

$$(2.3)\quad \begin{aligned} H^{\rm int}(\varphi) &= -g\sigma_i\tau_j\sqrt{\frac{4\pi}{3}}\int_0^\infty r^2 dr\,\frac{d\varrho}{dr}\,\varphi_{ij} = \sigma_i\tau_j u_{ij}(\varphi) \\ u_{ij}(\varphi) &= -g\sqrt{\frac{4\pi}{3}}\int_0^\infty r^2 dr\,\frac{d\varrho}{dr}\,\varphi_{ij}, \end{aligned}$$

where

$$(2.4)\quad \omega_l\left(r,\frac{d}{dr}\right) = c\left[-\frac{1}{r^2}\frac{d}{dr}\,r^2\frac{d}{dr} + \frac{l(l+1)}{r^2} + \mu^2\right]^{1/2}$$

$$(2.5)\quad \begin{aligned} &\left.\begin{matrix}\varphi_{1j}(r)\\ \varphi_{2j}(r)\end{matrix}\right\} = \frac{1}{i}\,\frac{\varphi_{1,1}^{(j)}(r) \pm \varphi_{1,1}^{(j)*}(r)}{\sqrt{2}} \\ &\varphi_{3j}(r) = \varphi_{1,0}^{(j)}(r), \\ &\left.\begin{matrix}\pi_{1j}(r)\\ \pi_{2j}(r)\end{matrix}\right\} = i\,\frac{\pi_{1,1}^{(j)}(r) \pm \pi_{1,1}^{(j)*}(r)}{\sqrt{2}} \\ &\pi_{3j}(r) = \pi_{1,0}^{(j)}(r) \end{aligned}$$

and

$$[\pi_{ij}(r),\varphi_{i'j'}(r')] = -i\hbar\delta_{ii'}\delta_{jj'}\,\frac{\delta(r-r')}{rr'}.$$

We have assumed above that $\varrho(\vec{x}) = \varrho(r)$.

We now use the expansion into a complete orthogonal system of radial functions $u_m(r)$ containing the function

$$u_1(r)=\frac{1}{C^{1/2}}\frac{d\varrho(r)}{dr}, \qquad C=\int_0^\infty r^2dr\left(\frac{d\varrho}{dr}\right)^2. \tag{2.6}$$

Then we have

$$\varphi_{ij}(r)=\sum_n q_{ijn}u_n(r)=q_{ij}\frac{1}{C^{1/2}}\frac{d\varrho(r)}{dr}+\sum_{n\neq 1}q_{ijn}u_n(r),$$

$$\pi_{ij}(r)=\sum_n p_{ijn}u_n(r)=p_{ij}\frac{1}{C^{1/2}}\frac{d\varrho(r)}{dr}+\sum_{n\neq 1}p_{ijn}u_n(r),$$

where

$$q_{ij}=q_{ijn}, \qquad p_{ij}=p_{ijn} \tag{2.7}$$

and

$$[p_{ijn}, q_{i'j'n'}]=-i\hbar\delta_{ii'}\delta_{jj'}\delta_{nn'}.$$

Thus, from (2.2) and (2.3) we get

$$\begin{aligned} H_1^M(\varphi,\pi)&=\sum_n\tfrac{1}{2}(p_{ijn}^2+\omega_{nn}^2q_{ijn}^2)+\sum_{n\neq n'}\tfrac{1}{2}\omega_{nn'}^2q_{ijn}q_{ijn'}=\\ &=\sum_{n\neq 1}\tfrac{1}{2}(p_{ijn}^2+\omega_{nn}^2q_{ijn}^2)+\sum_{\substack{n\neq n'\\ n>n'>1}}\tfrac{1}{2}\,\omega_{nn'}^2q_{ijn}q_{ijn'}+\\ &\quad+\tfrac{1}{2}(p_{ij}^2+\omega^2q_{ij}^2)+\sum_{n\neq 1}\omega_{n1}^2q_{ijn}q_{ij}, \end{aligned} \tag{2.8}$$

$$H^{\text{int}}(\varphi)=-g\sigma_i\tau_j\sqrt{\frac{4\pi}{3}}\left[\int_0^\infty r^2dr\left(\frac{d\varrho}{dr}\right)^2\right]^{1/2}q_{ij}=\sigma_i\tau_ju_{ij}(\varphi), \tag{2.9}$$

$$u_{ij}(\varphi)=-g\sqrt{\frac{4\pi}{3}}\left[\int_0^\infty r^2dr\left(\frac{d\varrho}{dr}\right)^2\right]^{1/2}q_{ij}=-\bar{g}\,q_{ij},$$

$$\bar{g}=g\sqrt{\frac{4\pi}{3}}\left[\int_0^\infty r^2dr\left(\frac{d\varrho}{dr}\right)^2\right]^{1/2},$$

where

$$\omega_{nn'}^2=\int_0^\infty r^2dr\,u_n(r)\,\omega_1^2\left(r,\frac{d}{dr}\right)u_{n'}(r),$$

$\omega^2=\omega_{11}^2$ (of course $\omega_{nn'}^2=\omega_{n'n}^2$).

From (2.9) we see that only nine oscillators $n=1$ of the pion field interact directly with the nucleon (the other oscillators n interact in-

directly with the nucleon, by means of oscillators $n=1$). It follows that the "potential" $V_\alpha(q)$ depends only on $q_{ij}=q_{ij1}$; $V_\alpha(\varphi)=V(q)$ $\left(\underline{u}_i(\varphi)=\underline{u}_i(q)\right)$.

We obtain $V(q)$ from (1.15) and (1.6), where $u_{ij}(\varphi)=-\bar{g}q_{ij}$. Also $H_1^{\text{int}}(\varphi)=H_1^{\text{int}}(q)$ and $H_2^{\text{int}}(\varphi)=H_2^{\text{int}}(q)$.

In equations (1.1) and (1.14) we shall treat as unperturbed Hamiltonians the operators (now $H^N=0$, $\rangle=\rangle^s\rangle^M$).

$$\overset{0}{H}=\sum_{l\neq 1}H_l^M(\varphi,\pi)+\sum_{n>1}\tfrac{1}{2}(p_{ijn}^2+\omega_{nn}^2q_{ijn}^2)+\sum_{n>n'>1}\omega_{nn'}^2q_{ijn}q_{ijn'}+\tfrac{1}{2}(p_{ij}^2+\omega^2q_{ij}^2)+H_1^{\text{int}}(q)$$

and

$$\overset{0}{H}_\alpha=\sum_{l\neq 1}H_l^M(\varphi,\pi)+\sum_{n>1}\tfrac{1}{2}(p_{ijn}^2+\omega_{nn}^2q_{ijn}^2)+ \\ +\sum_{n>n'>1}\omega_{nn'}^2q_{ijn}q_{ijn'}+\tfrac{1}{2}(p_{ij}^2+\omega^2q_{ij}^2)-V_\alpha(q). \tag{2.10}$$

The eigenvalue problem for H^0 or H^0_α gives eigenvalues

$$E_{\alpha A a N_2 N_3\ldots}=\sum_{l\neq 1}E_{N_l}+E_a+E_{\alpha A}$$

and eigenfuctions ($\rangle=\rangle^s\rangle^M$)

$$\Phi_{\alpha A a N_2 N_3}=|\alpha\rangle^s\prod_{l\neq 1}\Phi_{N_l}\cdot\Phi_a\cdot\Phi_{\alpha A},$$

where

$$\begin{cases} H_l^M(\varphi,\pi)\Phi_{N_l}=E_{N_l}\Phi_{N_l} \\ \left[\sum_{n\neq 1}\tfrac{1}{2}(p_{ijn}^2+\omega_{nn}^2q_{ijn}^2)+\sum_{n>n'>1}\omega_{nn'}^2q_{ijn}q_{ijn'}\right]\Phi_a=E_a\Phi_a \end{cases} \tag{2.11}$$

and

$$[\tfrac{1}{2}(p_{ij}^2+\omega^2q_{ij}^2)+V_\alpha(q)]\Phi_{\alpha A}=E_{\alpha A}\Phi_{\alpha A}. \tag{2.12}$$

We shall treat as the perturbation of $\overset{0}{H}$ the Hamiltonian

$$\overset{1}{H}=\sum_{n>1}\omega_{n1}^2q_{ijn}q_{ij}+H_2^{\text{int}}(q).$$

Its first term represents the interaction between nine oscillators $n=1$ and all other oscillators, $n\neq 1$, corresponding to the angular momentum $l=1$. This interaction causes transitions inside the same set a. The second term represents the interaction between nine oscillators $n=1$ and the nucleon, causing transitions between the different sets α. The probabilities of these transitions are given by products of both terms of $\overset{1}{H}$, because H_2^{int} alone cannot emit or absorb mesons $n\neq 1$. Thus, if matrix elements of $H_2^{\text{int}}(q)$ are small, these probabilities are distinctly

smaller than that of transitions inside the same set α, given by the first term of $\overset{1}{H}$. In this case one may speak of "fast" and "slow" transitions in the theory. As is well known, the oscillators corresponding to all $l \neq 1$ do not interact with the nucleon nor with the oscillators $l=1$ [1].

First equation (2.11) represents, of course, the eigenvalue problem for a system of independent oscillators. Second equation (2.11) describes the system of all oscillators $n \neq 1$ interacting mutually by means of the interaction $\sum\limits_{n>n'>1} \omega^2 nn' \, q_{ijn} \, q_{ijn'}$. The equation (2.12) describes nine oscillators $n=1$ interacting mutually by means of the potential $V_\alpha(q)$. One may try to consider these nine oscillators as an approximate model for the isobar α of the physical nucleon in the fixed source theory. These oscillators represent in our approximation the meson cloud surrounding the bare nucleon.

The results of this paper are dependent on the smallness of matrix elements of $H_2^{\text{int}}(q)$. So far, no exact solution of the equation (2.12) has been obtained which would enable us to investigate the smallness of matrix elements of $H_2^{\text{int}}(q)$. An approximation procedure will be given in a subsequent paper.

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CHEMISTRY

Thermodiffusion in Multi-Component Liquid Mixtures

by

B. BARANOWSKI and A. FULIŃSKI

Presented by M. ŚMIAŁOWSKI on September 18, 1956

In two-component aqueous solutions of electrolytes thermodiffusion produces, in general, an enrichment of the solution in the colder part of the system. The addition of another electrolyte, however, may cause a considerable change in Soret coefficients, including an inversion of the enrichment. This implies that the given electrolyte which migrates in a two-component solution towards regions of lower temperature may be made to migrate towards regions of higher temperature by the addition of another electrolyte. This, which is called electrolytic thermodiffusional inversion, was observed for the first time by L. J. Gillespie and S. Breck [1] and, independently, by K. Hirota [2].

Hitherto, the problem has been assessed theoretically [3]—[6] only with reference to ideal solutions of electrolytes, and not all the effects observed have been accounted for [7]. Nor has a description been offered of the changes observed [8] in thermodiffusional separation by addition of another component to mixtures of neutral molecules.

The purpose of the present work is to find a formula expressing the Soret partial coefficients of a multi-component liquid mixture. This refers to non-ideal systems of electrolyte as well as to non-electrolyte solutions. The arguments presented are based on Onsager's formulation of thermodynamics of irreversible processes according to the Alexander schema [9]—[11].

The Alexander equation [9] is taken as the starting point for the Soret equilibrium:

$$\sum_{j=1}^{N}\left(\frac{\varrho_j}{\varrho}\omega_j-\delta_{ij}\omega_i\right)(\bar{\mathbf{X}}_j+\bar{Q}_j^*\mathbf{X}_w)=0 \qquad (i=1,2,\ldots,N) \tag{1}$$

$$\delta_{ij}=\begin{cases}1 \text{ for } i=j\\ 0 \text{ for } i\neq j,\end{cases}$$

$\varrho=\sum_{i=1}^{N}\varrho_i$, where N denotes the number of components of the system, ω_i are "kinetic mobilities" connected with diffusion and self-diffusion constants [9], [10], [12], $\bar{Q}_i^*$ are "kinetic heats of transfer" linked with activation energies, and ϱ_j is the partial density of the j-th component.

Thermodynamic forces are:

$$\bar{X}_j=-(\operatorname{grad}\bar{\mu}_j)_{T,p}-z_jF\operatorname{grad}\varphi \tag{2}$$

(forces conjugated with mass fluxes),

$$X_W=-\frac{1}{T}\operatorname{grad}T \tag{3}$$

(force conjugated with the flux of heat),

where z_j stands for valency, $\bar{\mu}_j$ for the molar chemical potential of the j-th component, F for Faraday's constant, and grad φ for the gradient of the thermodiffusion potential. If the case concerns a mixture of molecules electrically neutral, grad φ is zero. For solutions of electrolytes, we assume for the solvent $z_N=0$. We can also increase the number of components whose $z_k=0$.

By transforming the equation (1), introducing the equations (2) and (3), applying the Gibbs-Duhem equation and the condition of electroneutrality, and by expressing chemical potentials as functions of concentrations and introducing Soret coefficients defined by

$$\left(\frac{D'}{D}\right)_k=-\frac{1}{x_k}\frac{\operatorname{grad}x_k}{\operatorname{grad}T}, \tag{4}$$

(where $(D'/D)_k$ stands for the Soret coefficient and x_k denotes the molar fraction of the k-th component), we obtain:

$$\sum_{j=1}^{N-1}\sum_{k=1}^{N-1}\left(\delta_{ij}+\frac{\omega_N}{\omega_i}\frac{x_j}{x_N}\right)\left(\frac{\partial\bar{\mu}_j}{\partial x_k}\right)_{T,p}^{*}x_k\left(\frac{D'}{D}\right)_k-\frac{z_i\Phi}{T}=\frac{1}{T}\bar{Q}_i \tag{5}$$

(where $\Phi=FT\dfrac{\operatorname{grad}\varphi}{\operatorname{grad}T}$ and $\bar{Q}_i=\bar{Q}_i^*-\dfrac{\omega_N}{\omega_i}\bar{Q}_N^*$ is the so-called phenomenological heat of transfer), and

$$\sum_{i=1}^{N-1}z_ix_i\left(\frac{D'}{D}\right)_i=0. \tag{6}$$

Equations (5) and (6) form a system of N linear non-homologous equations with a characteristic determinant other than zero, combining

N unknowns. For a system consisting of electrically neutral components $\Phi=0$, $z_i=0$, and we then have a system of $N-1$ equations (5) with $N-1$ unknowns. Solving the system of equations (5) for $(D'/D)_k$, and the system of equations (5) and (6) for Φ, we obtain

$$\left(\frac{D'}{D}\right)_i = \frac{\sum_{j=1}^{N-1}(-1)^{i+j}V_j^i\bar{Q}_j}{TV} + \frac{\sum_{j=1}^{N-1}(-1)^{i+j}V_j^i z_j}{TV}\cdot\Phi \qquad (i=1,2,\ldots,N-1) \tag{7}$$

$$\Phi = -\frac{\sum_{j=1}^{N-1}\sum_{k=1}^{N-1}(-1)^{j+k}V_j^k z_k x_k \bar{Q}_j}{\sum_{j=1}^{N-1}\sum_{k=1}^{N-1}(-1)^{j+k}V_j^k z_k x_k z_j}. \tag{8}$$

In these formulae we have the following determinants:

$$V_i^j = \mathrm{Det}\,|A_{p\neq i}^{q\neq j}| \qquad (p,q=1,2,\ldots,N-1),$$

and

$$A_p^q = \sum_{r=1}^{N-1}\left(\delta_{pr}+\frac{\omega_N}{\omega_p}\frac{x_r}{x_N}\right)\left(\frac{\partial\bar{\mu}_r}{\partial x_q}\right)^*_{T,P}x_q;\quad \left(\frac{\partial\bar{\mu}_r}{\partial x_q}\right)^*_{T,P} = \left(\frac{\partial\bar{\mu}_r}{\partial x_q}\right)_{T,P} - \left(\frac{\partial\bar{\mu}_r}{\partial x_N}\right)_{T,P}.$$

Formula (7) is a general formula for the Soret coefficient of the i-th component (of a neutral molecule or ion) in an arbitrary real liquid mixture containing electrically neutral substances as well as ions. For dilute ideal solutions of electrolytes, we obtain from the equation (7) the expression already obtained by de Groot [5] and from it the formulae of Norton [3] and Karger [6]. For a mixture of two electrically neutral liquids, we obtain the formula of Alexander [10]. For a mixture of three electrically neutral liquids (1-2-N), assuming that $x_i \ll x_N$, $x_N \approx 1$ $(i=1,2)$, we obtain

$$\left(\frac{D'}{D}\right)^*_1 = \frac{\left(\frac{\partial\bar{\mu}_1}{\partial x_1}\right)^*_{T,P}\left(\frac{\partial\bar{\mu}_2}{\partial x_2}\right)^*\left(\frac{D'}{D}\right)_1 - \frac{x_2}{x_1}\left(\frac{\partial\bar{\mu}_1}{\partial \mu_2}\right)^*_{T,P}\left(\frac{\partial\bar{\mu}_2}{\partial x_2}\right)_{T,P}\left(\frac{D'}{D}\right)_2}{\left(\frac{\partial\bar{\mu}_1}{\partial x_1}\right)^*_{T,P}\left(\frac{\partial\bar{\mu}_2}{\partial x_2}\right)^*_{T,P} - \left(\frac{\partial\bar{\mu}_1}{\partial x_2}\right)^*_{T,P}\left(\frac{\partial\bar{\mu}_2}{\partial x_1}\right)^*_{T,P}}, \tag{9}$$

and analogically for $(D'/D)^*_2$. $(D'/D)_1$ and $(D'/D)_2$ denote here Soret coefficients in the systems 1-N and 2-N respectively. Thus, the equation (9) indicates how the Soret coefficient will change in a given liquid as a result of the addition of a foreign component. These differences concern the derivatives of chemical potentials of the solutions in relation to their concentrations; in addition to the symmetrical derivatives $(\partial\bar{\mu}_i/\partial x_i)_{T,P}$, there also appear non-symmetrical ones: $(\partial\bar{\mu}_i/\partial x_j)_{T,P}$ $(i\neq j)$.

It follows from the equation (9) that, by taking a suitable additional component of the mixture, we can (in view of its value) improve the thermodiffusional separation of the component separated and if, in the given two-component system, separation fails to occur, i. e., $(D'/D)_1 = 0$, we can even bring about separation of the first substance by adding another component for which $(D'/D)_2 \neq 0$. Thus, the equation (9) may prove of considerable practical importance in separations of liquid organic mixtures.

For a dilute solution of one binary electrode, we obtain from the equations (7) and (8) for the thermodiffusion potential and the Soret coefficient, the expressions of Haase [13] and Alexander [10], by introducing the mean chemical potential and the mean Soret coefficient of the electrolyte.

It follows from the formulae obtained that the derivatives of chemical potentials in relation to concentrations, the symmetrical derivatives $(\partial\overline{\mu}_i/\partial x_i)_{T,P}$ and also the non-symmetrical ones $(\partial\overline{\mu}_i/\partial x_i)$ $(i \neq j)$ are, in particular, of basic importance. The value of the latter determined, as can be seen from the equation (9), whether thermodiffusion inversion is possible. If the thermodiffusion potential is disregarded, the equation (9) might be applied also to a solution consisting of two electrolytes and a neutral solvent. Even under this assumption, the three cases of Soret coefficient changes in three-component solutions of electrolytes would be expressed in relation to two-component solutions by the equation (9). Consideration of the thermodiffusion potential involves highly complicated formulae. For lack of space we have to refrain from quoting them here.

It has been shown that the equations (7) and (8) include all formulae hitherto developed for more simple cases. However, in order to obtain the formulae quoted by de Groot and Haase, or formulae based on their schema, it was necessary to assume that the solutions were dilute. Neither de Groot nor Haase achieved this directly in their formulations. They chose, however, as a system of reference, an *immobile* quasi-crystal lattice of a liquid, while the schema used in deriving equations (7) and (8) was that of Alexander in which a *mobile* quasi-crystal lattice is used for the same purpose. Consequently, it appears that the adoption of an immobile system of reference is justified only in cases of dilute solutions, where a change in the concentration of the substance dissolved in an element of volume fails in practice to change the concentration of the solvent — i. e., when the "recoil velocity" is small.

This theory indicates the limits of possible interpretations afforded by Onsager's approach to multicomponent thermodiffusion. The basic assumption is that partial heats of transfer do not depend on the composition of the system (which corresponds to the assumption of inva-

riable phenomenological coefficients). Taking into account these changes, and assessing the influence on irreversible effects of inappreciable foreign additions, appears to be possible on the basis of the thermodynamics of irreversible processes as assessed by Reik [14].

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CHEMISTRY

Absorption Spectrum of Monocrystalline Resorcinol Plates in the Near Ultraviolet

by

J. ROHLEDER

Presented by M. ŚMIAŁOWSKI on September 26, 1956

Although the ultraviolet absorption spectrum of resorcinol (meta-dihydroxybenzene) has been known for long, both in solution [1]—[5] and in gaseous state [6], [7], absorption in solid state has not yet been investigated.

The ultraviolet absorption spectrum of resorcinol vapour is rather complicated, however, according to Beck [7], there may be distinguished three groups of bands whose "violet" edges are at the following values of wave numbers:

$$35{,}946, \quad 36{,}663 \quad \text{and} \quad 36{,}912\ \text{cm}^{-1}.$$

The absorption spectrum of resorcinol solution is much simpler; it consists of a single broad band the peak of which is, in the case of aqueous solutions, in the neighbourhood of 274 mμ. [1]—[5]. The band is attributed to nonionized resorcinol molecules. H. H. Hodgson [5] believes that the absorption curve represents the envelope of the fine structure which fails to come into evidence in solutions.

In this paper we have studied the absorption spectrum of monocrystalline resorcinol plates in the near ultraviolet. Considering the very high absorption coefficient within this range of the spectrum, very thin plates had to be used. The plates were made *) of three times sublimated β-resorcinol by prolonged heating in a sand bath. The plates were perfectly transparent within the visible range of the spectrum. They were between 0.02 and 0.1 mm. thick and averaged 5×12 mm. in size. The samples were fixed in small windows in metal screens and their transparency (intensity of light — I) was compared with that of uncovered

*) The author is greatly indebted to A. Gołębiowski for producing the plates used in the experiments.

windows (intensity of light — I_0). Losses due to reflection were disregarded.

Transparency measurements were carried out with the aid of photoelectric photometry methods using a high-pressure mercury lamp, HQE 40, and a Zeiss monochromator with a rock-salt prism. The photoelectric current was measured with the aid of Wulf's electrometer.

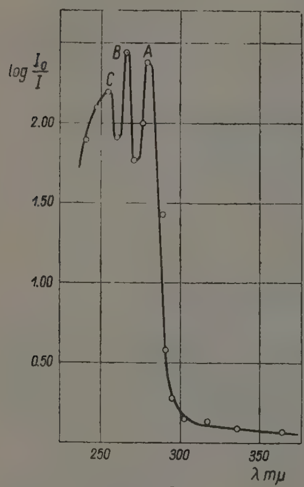

Fig. 1

The results obtained from one of the plates examined are recorded in Fig. 1. Owing to considerable errors in determining the thickness of the plates the values of log (I_0/I), instead of the absorption constant, were marked on the axis of ordinates. The wave-length was marked on the axis of abscissae. The plate was approximately 0.1 mm. thick.

Beginning with 300 mμ, absorption of resorcinol in solid state grows rapidly with decreasing wave-length, showing three maxima (absorption bands A, B, C), and begins to drop at about 250 mμ. The positions of the maxima of the three bands are shown in the table below:

absorption band	λ max in mμ	corresponding energy in eV
A	281 ± 3	4.41 ± 0.05
B	265 ± 3	4.68 ± 0.05
C	253 ± 1	4.90 ± 0.02

On the long-wave side, the wave-length corresponding to half of the ordinate of band A is

$$\lambda_{1/2} = 289 \pm 2 \text{ m}\mu.$$

Reckoned in terms of energy the value amounts to

$$E_{opt} = 4.29 \pm 0.02 \text{ eV},$$

and represents the optical energy of activation of monocrystalline resorcinol plates.

Thus, in the case of solid resorcinol, there appears an absorption band structure (3 peaks) which fails to come into evidence in the case of solutions. In addition, it may be noted that the absorption band A (281 ± 3 mμ) of solid resorcinol is slightly shifted towards the longer waves in comparison to the corresponding band observed in resorcinol

vapour by Beck. In comparison to the absorption spectrum of resorcinol aqueous solutions (maximum in the neighbourhood of 274 mμ.) the shift is still more distinct.

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CHEMISTRY

Theoretical Considerations on the Possibility of a New Method for Separating Mixtures and Isotopes

by

A. WITKOWSKI

Presented by B. KAMIEŃSKI on October 2, 1956

1. In the present considerations it is assumed that vapour rising from a boiling liquid mixture passes between two vertical plates or two concentric tubes fixed over the liquid. The temperature of one of the two plates (or tubes) is regulated so as to cause condensation of vapour, whereas the temperature of the other plate is higher. Furthermore, it is assumed that, in addition to the vessel with the boiling liquid, there is another vessel above the plates (tubes) which is not cooled and in which the gaseous phase can be collected.

The horizontal temperature gradient causes thermal diffusion in the gaseous phase producing thereby a flow of mass which together with the convection current in the gaseous phase result in turn in concentration changes in both reservoirs. However, at the same time a film of liquid is formed on the "cool" wall. This film is in motion in respect to the whole gaseous phase and as a result distillation takes place on the wall. This process is superimposed on the process of separation by thermal diffusion. If both thermal diffusion and distillation have one direction, which usually is the case in isotopic mixtures and in many other mixtures, the movements of both phases will be accompanied by the adding up and the intensification of the concentration effects at every elementary stage of the process. It is to be stressed that even if in the proposed method no distillation takes place, i. e., if the relative velocity is one, the thermal diffusion process will proceed, and the conditions will still be much more favourable than in the thermogravitational method used at present. Indeed, owing to the presence of liquid on the cooled wall the flow of mass caused by thermal diffusion may transfer the enriched constituent from the wall to the gaseous phase.

2. Let us now consider the theoretical basis of the method. The first step will be to introduce the Cartesian arrangement of co-ordinates.

With this arrangement the Z axis is vertical and the X axis is normal to the surface of the two plates. The breadth of the plates is b and the distance between them is a. The origin of the system is at the hotter plate. A binary mixture is being considered.

Then, for the "lighter" constituent, i. e., the one enriched through distillation in the gaseous phase and migrating to the hotter plate, the continuity equation is:

$$\frac{\partial(\varrho c)}{\partial t} + div\hat{J} = 0, \tag{1}$$

where the components of the vector of the flow of mass are

$$J_x = -D\varrho\frac{\partial c}{\partial x} + D\alpha c(1-c)\frac{1}{T}\frac{\partial T}{\partial x}\varrho, \tag{2}$$

$$J_z = -D\varrho\frac{\partial c}{\partial z} + v\varrho c, \tag{3}$$

and c is the concentration, D the diffusion coefficient, α the thermal diffusion constant, T temperature, ϱ density and v the velocity of convection current.

Boundary conditions for equations (1) are as follows:

$$\lim_{x\to 0} J_x = 0, \tag{4}$$

$$\lim_{x\to\xi-}\frac{c}{1-c} = \varepsilon \lim_{x\to\xi+}\frac{c_l}{1-c_l}, \tag{5}$$

where c_l is the concentration in the liquid phase, ξ the distance from the hotter wall to the surface of the liquid, and ε is the relative volatility.

The velocity of convection currents in (3) is calculated from hydrodynamic equations, taking into account temperature changes of density. In the case under consideration where the movements are one-dimensional, these equations become linear. The boundary conditions accepted for the convection current are as follows: velocity equals zero for $x=0$, the continuity of the tangent component in the transition from liquid to gaseous phase and the lack of flow of the mixture through the horizontal section of the apparatus. From the last condition it follows that then flowing occurs through the gaseous phase and is defined by the following equation:

$$b\int_0^{\xi} v\varrho\, dx = b\delta. \tag{6}$$

To calculate the significant values the non-linear equation (1) with variable coefficients is used in the following manner. It is assumed [1], [2] that the horizontal gradient of concentration is not dependent on height, and that between the plates the process is quasi-stationary. The func-

tion $G(x,z)$ is now introduced; it is defined by equation

$$\frac{\partial c}{\partial z}G = J_x. \tag{7}$$

It can be demonstrated that this function exists and that it depends only on the co-ordinate x. Moreover, it can be shown that this function satisfies the equation

$$\frac{d^3}{dx^3}\left(\frac{1}{\varrho}\frac{dG}{dx}\right) + \frac{g\varrho\Delta T}{\mu\xi T} = 0, \tag{8}$$

where g is the acceleration of gravity, ΔT the difference of temperatures between the plates, T the mean temperature and μ is the viscosity. The boundary conditions of function G are

$$\left.\begin{aligned} G(0) &= 0,\\ G(\xi) &= -\delta,\\ \left(\frac{dG}{dx}\right)_{x=0} &= 0,\\ \left(\frac{dG}{dx}\right)_{x=\xi} &= -\varkappa\varrho, \end{aligned}\right\} \tag{9}$$

where $\varkappa$ is the velocity of the surface of the liquid.

The mass of the lighter constituent m flowing in a time unit through the horizontal section of the apparatus is defined by the equation

$$m = \int_0^{\xi}\int_0^{b} J_z\,dx\,dy + \int_{\xi}^{a}\int_0^{b} J_z^l\,dx\,dy = \tag{10}$$

$$= -bD\int_0^{\xi}\varrho\frac{\partial c}{\partial z}\,dx + b\int_0^{\xi}c\varrho v\,dx + b\int_{\xi}^{a}c_l\varrho_l v_l\,dx - bD_l\int_{\xi}^{a}\frac{\partial c_l}{\partial z}\varrho_l\,dx,$$

where the index l attached to a symbol means that the symbol corresponds to the liquid phase. When it is remembered that the concentration changes slowly with x, the equation for the relationship between the concentration and height is obtained by using the properties of function G:

$$m = -\frac{dc}{dz}\left[Db\varrho\xi + \frac{b}{D\varrho}\int_0^{\xi}G^2dx\right] + c(1-c)\left[b\,\delta\ln\varepsilon + \frac{a\Delta Tb}{\xi T}\int_0^{\xi}|G|\,dx\right] = \tag{11}$$

$$= -H\frac{dc}{dz} + Kc(1-c).$$

The expressions for the values H and K obtained after simplifying are as follows:

$$H = b\delta\ln\varepsilon + \frac{a(\Delta T)^2 g\varrho^2}{6!\,\mu T^2}\xi^3 b + \frac{a\Delta T}{2T}b\delta \tag{12}$$

$$K = Db\varrho\xi + \frac{(\Delta T)^2 g^2\varrho^3}{9!\,D\mu^2T^2}\xi^7 b + \frac{g\varrho\Delta T}{6!\,\mu DT}\xi^4 b\delta + \frac{13}{35D\varrho}b\xi\delta^2. \tag{13}$$

3. If in the apparatus here described the amount of liquid distilled in a time unit gradually decreases, then in the limit case $\delta = 0$, i. e., when the liquid disappears from the colder plate, the apparatus ought to behave as a thermal diffusion column.

Indeed, for $\delta \to 0$, the limiting case for the formulae (12) and (13) is represented by the formulae deduced by Furry, Jones and Onsager [1], [2] in the well-known theory of the thermogravitational method.

On the other hand, in the second limiting case, for $\Delta T = 0$, the apparatus will behave like a rectifying column.

Now, by assuming $\Delta T \to 0$ and the state of vertical equilibrium ($m = 0$) it is easy, from formulae (11), (12) and (13), to obtain the formula for the height equivalent to the theoretical plate as deduced by Westhaver [3] in his theory of the rectifying column.

4. In order to establish to what extent one can expect that the separation of isotopic mixtures with the new method will surpass, under similar conditions, the separation by thermogravitation or distillation, a numerical example will be considered. For computations the following average values are accepted: $\varepsilon = 1.02$, $a = 6 \cdot 10^{-2}$, $b\delta = 3 \cdot 10^{-3}$ g./sec, $\xi = 5 \cdot 10^{-1}$ cm., $\Delta T = 300°$, $T = 523°$ K, $D = 5 \cdot 10^{-1}$ cm²/sec, $b = 3$ cm., $\varrho = 6 \cdot 10^{-4}$ g./cm³, $\mu = 3 \cdot 10^{-4}$ poise. From computations based on the above theoretical considerations it follows that in the average case the height equivalent of the theoretical plate with the new method is only half of the height obtained under similar conditions in isotope separation by thermogravitational techniques. In this connection let us consider a case when in a column 1.5 m. high there is a state of equilibrium and the concentration is $c_i \ll 1$. Then in the numerical example the concentration ratio in the reservoirs will be 20 times greater in the proposed apparatus than the separation obtained under similar conditions by the thermogravitational method, and 11 times greater than separation by distillation.

A fuller report on the method will be published shortly in Acta Physica Polonica.

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CHEMISTRY

The Potential of the Adsorption Micro-electrode in Air Contaminated by Oxidising Gases

by

B. KAMIEŃSKI and J. KULAWIK

Presented by B. KAMIEŃSKI on October 2, 1956

The adsorption micro-electrode described in former papers [1]—[8] was used as a sensitive detector of halogens or of reducing gases in air [3]—[15]. The electrode potential rises when the concentration of the oxidising agent is increased, and sinks with decreasing concentration. As a consequence of this behaviour of the electrode, it may be expected that the potential of the electrode would decrease if oxygen were replaced by nitrogen in the atmosphere.

A Wulf electrometer was used to measure the potential of the adsorption micro-electrode composed of a thin platinum wire, the cross-section of which was covered by a microscopic film of silica gel. This adsorption layer was exposed to the surrounding atmosphere.

The oxidising action of oxygen was determined by preparing different mixtures of nitrogen with oxygen and exposing the hanging adsorption micro-electrode, which changed its potential when different mixtures were applied. First of all, air was replaced by pure nitrogen. Measurements of the potential with time are represented in Fig. 1. The arrow in Fig. 1 denotes the first change of the potential with time when pure nitrogen was applied instead of air. When the potential became stable, a mixture of 99% nitrogen and 1% (volume) oxygen was applied. The potential rose by steps when 1%, 10% and 100% oxygen mixtures were used as represented by arrows. The potential of the electrode became somewhat

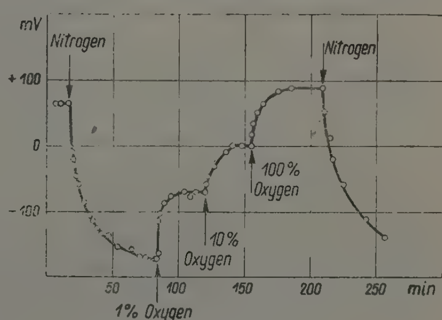

Fig. 1

higher in 100% oxygen than in air (Fig. 1). The last asymptotic decrease of the potential in Fig. 1 was observed when pure oxygen was replaced by nitrogen.

As may be seen in Fig. 1, oxygen has a strong influence on the potential of the electrode. The behaviour of the electrode became more interesting when air was contaminated by reducing agents, and the ac-

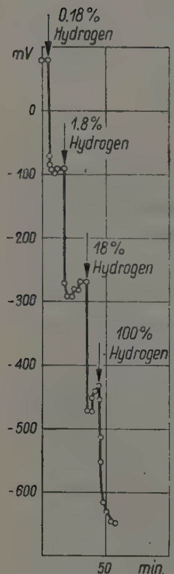

Fig. 2

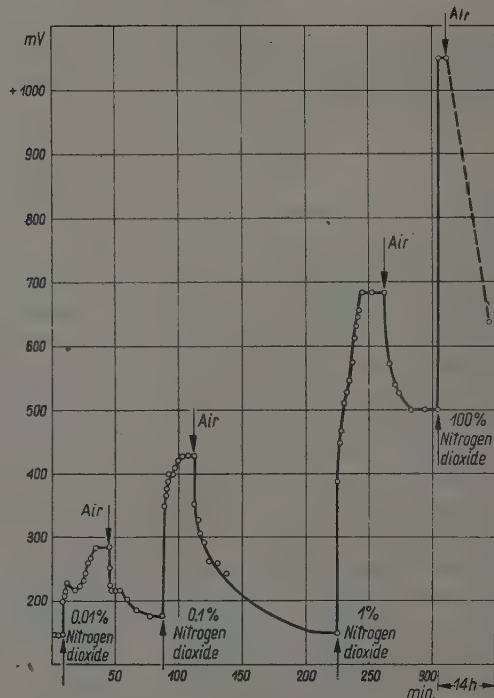

Fig. 3

tion of oxygen became weaker. The reducing capacity of hydrogen was investigated; it was much stronger when no oxygen was present in the surrounding atmosphere. Nevertheless, the action of hydrogen was sufficiently strong in air even when small concentrations of hydrogen were present. Fig. 2 represents the change of the potential (ordinate) with increasing quantities of hydrogen (arrows) with time (abscissa). As may be seen in Fig. 2, the presence of 0.18% hydrogen in air was accompanied by a potential change of 178 mV. However, when air was replaced by a mixture of nitrogen (devoid of O_2) with 0.18% hydrogen, then the potential change was 620 mV, as may be seen in Fig. 4, curve II. On the same diagram, the potential changes of the micro-electrode are

shown when different concentrations of hydrogen in nitrogen were applied. Comparison of diagrams 2 and 4 shows that hydrogen acted strongly on the electrode, and when no oxygen was present, some tenths of one per cent of hydrogen sufficed to saturate the platinum electrode with hydrogen (Fig. 4, curves I and II). The saturation with hydrogen was evident at low concentrations because further increases in hydrogen concentration did not appreciably change the potential. The high diffusion velocity of hydrogen enabled the oxygen action on the electrode to be quite efficiently intercepted. On the other hand, oxygen acted on hydrogen adsorbed at the platinum silica gel interface and bound it, to give water molecules. The resulting potential may be treated as an implicit function of the diffusion constant, and of the rate of building water molecules at the interface.

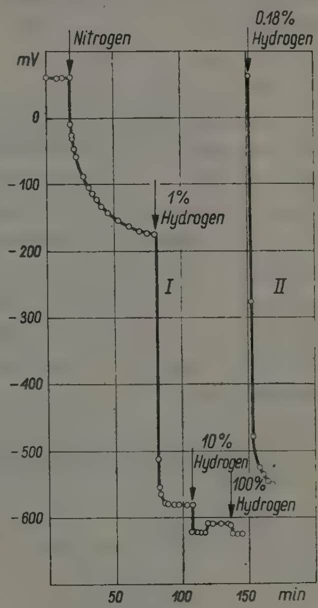

Fig. 4

During a discussion on the properties of the micro-electrode, Professor W. Bobrownicki suggested an investigation concerning electrode behaviour when nitrogen oxides contaminate the air, as frequently happens in the neighbourhood of factories. We prepared nitrogen dioxide by thermal decomposition of lead nitrate, and, after liquefaction and purification, made suitable mixtures of air and nitrogen dioxide. The smallest concentration, the preparation of which was to some degree justified, was 0.01% nitrogen dioxide in air. The real concentration was probably smaller because of the adsorption on the glass walls. Nevertheless, even such traces of nitrogen dioxide changed the potential in the positive direction, to c. 130 mV. When the mixture was displaced again by air, the potential decreased asymptotically to c. 100 mV (Fig. 3). The decrease was not exactly reversible. However, the potential change indicates in the light of former publications that a twofold decrease of the potential was accompanied by approximately a tenfold decrease of the concentration. The proportion of the potential changes was $130:30 = 2^x$, where x is somewhat greater than 2; this means that the decrease in 100 mV (from 130 to 30) corresponded to a more than hundredfold decrease of the concentration. This rough estimate leads to the conclusion that the active concentration of nitrogen dioxide on the electrode was approximately 0.0001%, or even less. It is evident that there are

considerable difficulties in removing such traces of nitrogen dioxide from the electrode. Fig. 3 represents the measurements of the electrode potential with time when different concentrations of nitrogen dioxide were applied. The curve indicates that it was possible to remove nitrogen dioxide from the electrode in a stream of fresh air when the concentration of oxide did not exceed 0.1%. When stronger mixtures were used, nitric acid was left on the electrode and it was impossible to displace it even after 14 hours, as may be seen by observing the last break of the curve in Fig. 3.

Atmospheric contaminations by nitrogen dioxide in factories are on the low side, and in these circumstances the action of the electrode may be reversible. An approximate estimate makes it possible to define the potential change of the electrode when the concentration of nitrogen dioxide is assumed as 0.001% in air. Taking into consideration that a twofold decrease of the potential change is accompanied by a tenfold decrease of the concentration, we find that the potential change would be c. 65 mV at 0.001% nitrogen dioxide in air. The sensibility of the electrode suggests a possible application of the device when ordinary methods fail.

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Extension of the Phenomenological Theory of the Electrothermodiffusion Method

by

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Soret coefficients of electrolyte solutions have been determined recently by the electrothermodiffusion method [1]—[4], which represents a modification of the thermogravitational method. The method permits the use of relatively simple means in determining the coefficient, on the basis of the phenomenological theory presented earlier [5], [6]. The present work seeks to provide an extension of the existing phenomenological theory, and thus to make it possible to calculate Soret coefficients with greater precision.

Gradient of temperature in a capillary

The equation of thermal conductivity (disregarding the effect of thermodiffusion on thermal conductivity) assumes for a stationary state the form

$$-div \underline{J}_q + \sigma_q = 0, \tag{1}$$

where $\underline{J}_q$ denotes the vector of the heat flow expressed on the basis of Fourier's law, whereas σ_q stands for the source of heat generated in the capillary by the flowing current:

$$\sigma_q = \frac{\lambda U^2 \varkappa'}{h_0^2}, \tag{2}$$

where λ stands for the electrical equivalent of heat, U for the potential applied to the capillary, h_0 for the height of the capillary, and $\varkappa'$ for the local specific conductance of the liquid. The last named value depends on the temperature and concentration of the electrolyte, both of

which are functions of the distance from the axis of the capillary:

$$d\varkappa' = \left(\frac{\partial \varkappa'}{\partial_t}\right) dt + \left(\frac{\partial \varkappa'}{\partial_n}\right) dn, \tag{3}$$

where t denotes the temperature and n the concentration of the electrolyte.

Expressing horizontal changes of concentration in the stationary state by the equation

$$dn = -\frac{D'}{D} n\, dt, \tag{4}$$

where D'/D stands for the Soret coefficient, we obtain for the source of heat the following equation after integration of equations (3) and (4):

$$\sigma_q = \frac{\lambda U^2}{h_0^2} [c_\varkappa(t-t') + \varkappa], \tag{5}$$

where $\varkappa$ denotes electrical conductance at the temperature t', which corresponds to the mean temperature prevailing in the capillary, and $c_\varkappa$ indicates

$$c_\varkappa = \left(\frac{\partial \varkappa'}{\partial t}\right) - \frac{D'}{D}\left(\frac{\partial \varkappa'}{\partial \ln n}\right). \tag{6}$$

In the existing phenomenological theory [5], [6] only the dependence of electrical conductance on temperature is accounted for. Since electrical conductance represents at low concentrations of the electrolyte an increasing function of the concentration, the heat source will be in agreement with formula (6), at a positive Soret coefficient, smaller than when only the dependence of electrical conductance on temperature is accounted for.

It can be demonstrated that the equation (1) can, taking into account the equation (5), be transformed into Bessel's equation, in the same way as in the case of the existing phenomenological theory [5], [6]. In this case, the constants "a" and "b" used in the existing theory [5], [6] will be substituted by the values

$$a' = \frac{\lambda U^2 c_\varkappa r_0^2}{h_0^2 k}, \tag{7}$$

$$b' = \frac{\lambda U^2 r_0^2}{h_0^2 k} \quad (\varkappa - c_\varkappa t'), \tag{8}$$

where r_0 stands for the inner radius of the capillary, and k for the coefficient of the thermal conductivity of the liquid. The distribution of temperature obtained from formula (5) changes into that quoted earlier [5], [6], for systems of a low Soret coefficient or slight dependence of electrical conductance on the concentration of the solute.

Flow of mass between reserve containers

Instead of the formula (4) used in the theory existing hitherto [5], [6], we shall now use, to express the concentration course, the following simplified equation:

$$dn = \frac{-D'}{D} n_0 dt, \tag{9}$$

where n_0 denotes the initial concentration of the electrolyte. The use of formula (9) is justified by the fact that the concentration changes induced by thermodiffusion in the level section of the capillary are insignificant and that, consequently, the thermodiffusion flux can be assumed to be proportional to the initial concentration. Using the formula expressing the temperature course we obtain, after integrating the equation (9), introducing the condition of mass preservation in the level section, the formula

$$n = n_0 \left\{ 1 + \frac{D'}{D} A \left[\frac{2J_1(\sqrt{a'})}{\sqrt{a'}} - J_0(u) \right] \right\}, \tag{10}$$

where J_0 and J_1 denote Bessel's function of the first kind, zero and first order, whereas A and U indicate

$$A = \frac{\lambda M}{2\pi h_0 \sqrt{a'}\, k J_1(\sqrt{a'})}; \qquad u = \sqrt{a'} \frac{r}{r_0},$$

where M stands for the power of the current, and r for the distance from the axis of the capillary. The distribution of the concentration as expressed by the equation (10) agrees within 1 per cent with the course obtained after integration of formula (4).

The relation between the concentrations in the upper and the lower reserve containers after the time τ, is expressed on the basis of formula (10):

$$\frac{n_D}{n_G} = 1 + \frac{4\pi\tau r_0^4 (\Delta t)^2 \left(\beta\varrho + \frac{D'}{D} n_0 a \right) \frac{D'}{D}}{V a'^2 \eta [1 - J_0(\sqrt{a'})]^2} \left\{ J_0^2(\sqrt{a'}) \left[\frac{a'}{2} - 4 \right] + \right.$$
$$\left. + J_1^2(\sqrt{a'}) \left[\frac{a'}{2} - \frac{16}{a'} \right] + J_0(\sqrt{a'}) J_1(\sqrt{a'}) \left[\frac{16}{\sqrt{a'}} - \sqrt{a'} \right] \right\}. \tag{11}$$

In the above formula, $\beta\varrho$ indicates the derivative of density in relation to temperature, a — the derivative of density in relation to concentration, V — the volume of the reserve container, η — the viscosity coefficient, Δt — the difference in temperature between the axis and the inner wall of the capillary expressed by the formula

$$\Delta t = A[1 - J_0 \sqrt{a'}]. \tag{12}$$

The equation (11) is inconvenient for calculating Soret coefficients. Therefore, using the rapid convergence of Bessel's functions, the relevant series may be cut after three expressions and substituted in integrating the formula for the ratio of concentrations. We obtain then

$$\frac{n_D}{n_G}=1+\frac{2\pi\tau r_0^4 g(\bar{\Delta}t)^2\left(\beta\varrho+\frac{D'}{D}n_0\alpha\right)\frac{D'}{D}\left(1-\frac{a'}{20}\right)}{576\eta V\left(1-\frac{a'}{8}+\frac{a'^2}{192}\right)^2}, \tag{13}$$

where

$$\bar{\Delta}t=\frac{\lambda M}{4\pi h_0 k} \tag{14}$$

indicates the maximum difference of temperatures in the capillary obtained when disregarding the dependence of electrical conductance on temperature and concentration.

Calculation of Soret coefficients

Soret coefficients may be calculated on the basis of formula (11) or (13).

The table below presents results for aqueous solutions of cupric sulphate calculated on the basis of results published earlier [1], [2]. The concentration is given in normalities. The first column comprises results calculated on the basis of the existing phenomenological theory [5], [6], whereas the second column contains Soret coefficients calculated with the aid of formula (13). In computing the values, the method of consecutive approximations was used; the Soret coefficient calculated without accounting for the dependence of electrical conductance on concentration was consistently used as zero approximation. The computations were limited to the first approximation since further computations differed from it only to a negligible degree (within the limit of experimental error).

Normality	$\frac{D'}{D}\times 10^3$ [1]	$\frac{D'}{D}\times 10^3$ (equation (13))
1	5.5	8.3
0.75	5.8	8.1
0.5	6.3	9.3
0.25	5.6	8.6
0.1	5.6	9.3
0.05	4.3	6.8

As the table shows, the results obtained with the aid of formula (13) are closer to the values obtained by other methods [1], [2] than those based on the hitherto existing phenomenological theory [5], [6].

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